Merkblatt zu Differentialoperatoren

Definition des Nabla-Operators in kartesischen Koordinaten:

$$\nabla = \boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_y \frac{\partial}{\partial y} + \boldsymbol{e}_z \frac{\partial}{\partial z} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}$$

Gradient:

$$\begin{aligned}
\operatorname{grad} V(\boldsymbol{r}) = \nabla V(\boldsymbol{r}) &= \boldsymbol{e}_x \frac{\partial V(\boldsymbol{r})}{\partial x} + \boldsymbol{e}_y \frac{\partial V(\boldsymbol{r})}{\partial y} + \boldsymbol{e}_z \frac{\partial V(\boldsymbol{r})}{\partial z} && \text{(kartesische Koordinaten)} \\
&= \boldsymbol{e}_\varrho \frac{\partial V(\boldsymbol{r})}{\partial \varrho} + \boldsymbol{e}_\varphi \frac{1}{\varrho} \frac{\partial V(\boldsymbol{r})}{\partial \varphi} + \boldsymbol{e}_z \frac{\partial V(\boldsymbol{r})}{\partial z} && \text{(Zylinderkoordinaten)} \\
&= \boldsymbol{e}_r \frac{\partial V(\boldsymbol{r})}{\partial r} + \boldsymbol{e}_\vartheta \frac{1}{r} \frac{\partial V(\boldsymbol{r})}{\partial \vartheta} + \boldsymbol{e}_\varphi \frac{1}{r \sin\vartheta} \frac{\partial V(\boldsymbol{r})}{\partial \varphi} && \text{(Kugelkoordinaten)}
\end{aligned}$$

Divergenz:

$$\begin{aligned}
\operatorname{div} \boldsymbol{A}(\boldsymbol{r}) = \nabla \cdot \boldsymbol{A}(\boldsymbol{r}) &= \frac{\partial A_x(\boldsymbol{r})}{\partial x} + \frac{\partial A_y(\boldsymbol{r})}{\partial y} + \frac{\partial A_z(\boldsymbol{r})}{\partial z} \\
&= \frac{1}{\varrho} \frac{\partial}{\partial \varrho} (\varrho A_\varrho(\boldsymbol{r})) + \frac{1}{\varrho} \frac{\partial A_\varphi(\boldsymbol{r})}{\partial \varphi} + \frac{\partial A_z(\boldsymbol{r})}{\partial z} \\
&= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r(\boldsymbol{r})) + \frac{1}{r \sin\vartheta} \frac{\partial}{\partial \vartheta} (A_\vartheta(\boldsymbol{r}) \sin\vartheta) + \frac{1}{r \sin\vartheta} \frac{\partial A_\varphi(\boldsymbol{r})}{\partial \varphi}
\end{aligned}$$

Ein Vektorfeld mit $\operatorname{div} \boldsymbol{A} = 0$ heißt „quellenfrei“.

Rotation:

$$\begin{aligned}
\operatorname{rot} \boldsymbol{A}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}(\boldsymbol{r}) &= \boldsymbol{e}_x \left[\frac{\partial A_z(\boldsymbol{r})}{\partial y} - \frac{\partial A_y(\boldsymbol{r})}{\partial z} \right] + \boldsymbol{e}_y \left[\frac{\partial A_x(\boldsymbol{r})}{\partial z} - \frac{\partial A_z(\boldsymbol{r})}{\partial x} \right] \\
&\quad + \boldsymbol{e}_z \left[\frac{\partial A_y(\boldsymbol{r})}{\partial x} - \frac{\partial A_x(\boldsymbol{r})}{\partial y} \right] \\
&= \boldsymbol{e}_\varrho \left[\frac{1}{\varrho} \frac{\partial A_z(\boldsymbol{r})}{\partial \varphi} - \frac{\partial A_\varphi(\boldsymbol{r})}{\partial z} \right] + \boldsymbol{e}_\varphi \left[\frac{\partial A_\varrho(\boldsymbol{r})}{\partial z} - \frac{\partial A_z(\boldsymbol{r})}{\partial \varrho} \right] \\
&\quad + \boldsymbol{e}_z \frac{1}{\varrho} \left[\frac{\partial}{\partial \varrho} (\varrho A_\varphi(\boldsymbol{r})) - \frac{\partial A_\varrho(\boldsymbol{r})}{\partial \varphi} \right] \\
&= \boldsymbol{e}_r \frac{1}{r \sin\vartheta} \left[\frac{\partial}{\partial \vartheta} (A_\varphi(\boldsymbol{r}) \sin\vartheta) - \frac{\partial A_\vartheta(\boldsymbol{r})}{\partial \varphi} \right] \\
&\quad + \boldsymbol{e}_\vartheta \frac{1}{r} \left[\frac{1}{\sin\vartheta} \frac{\partial A_r(\boldsymbol{r})}{\partial \varphi} - \frac{\partial}{\partial r} (r A_\varphi(\boldsymbol{r})) \right] \\
&\quad + \boldsymbol{e}_\varphi \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\vartheta(\boldsymbol{r})) - \frac{\partial A_r(\boldsymbol{r})}{\partial \vartheta} \right]
\end{aligned}$$

Ein Vektorfeld mit $\operatorname{rot} \boldsymbol{A} = 0$ heißt „wirbelfrei“.

Laplace:

$$
\begin{aligned}
\Delta V(\boldsymbol{r}) = \nabla^2 V(\boldsymbol{r}) &= \frac{\partial^2 V(\boldsymbol{r})}{\partial x^2} + \frac{\partial^2 V(\boldsymbol{r})}{\partial y^2} + \frac{\partial^2 V(\boldsymbol{r})}{\partial z^2} \\
&= \frac{1}{\varrho}\frac{\partial}{\partial \varrho}\left(\varrho \frac{\partial V(\boldsymbol{r})}{\partial \varrho}\right) + \frac{1}{\varrho^2}\frac{\partial^2 V(\boldsymbol{r})}{\partial \varphi^2} + \frac{\partial^2 V(\boldsymbol{r})}{\partial z^2} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V(\boldsymbol{r})}{\partial r}\right) + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial V(\boldsymbol{r})}{\partial \vartheta}\right) + \frac{1}{r^2 \sin^2\vartheta}\frac{\partial^2 V(\boldsymbol{r})}{\partial \varphi^2}
\end{aligned}
$$

In beliebigen krummlinigen Koordinaten: In den Koordinaten u_1, u_2 und u_3 zur Darstellung des Vektors $\boldsymbol{r} = \boldsymbol{r}(u_1, u_2, u_3)$ erhält man mit den Basisvektoren $\boldsymbol{e}_i = \frac{1}{h_i}\frac{\partial \boldsymbol{r}}{\partial u_i}$ und mit $h_i = \left|\frac{\partial \boldsymbol{r}}{\partial u_i}\right|$ für das Skalarfeld $V(u_1, u_2, u_3)$ und das Vektorfeld $\boldsymbol{A}(u_1, u_2, u_3) = \sum_i A_{u_i}(u_1, u_2, u_3)\boldsymbol{e}_i(u_1, u_2, u_3)$ die folgenden Darstellung der Differentialoperatoren:

$$
\begin{aligned}
\operatorname{grad} V = \nabla V &= \sum_i \boldsymbol{e}_i \frac{1}{h_i}\frac{\partial V}{\partial u_i} \\
\operatorname{div} \boldsymbol{A} = \nabla \cdot \boldsymbol{A} &= \frac{1}{h_{u_1} h_{u_2} h_{u_3}}\left[\frac{\partial}{\partial u_1}(h_{u_2} h_{u_3} A_{u_1}) + \frac{\partial}{\partial u_2}(h_{u_1} h_{u_3} A_{u_2}) + \frac{\partial}{\partial u_3}(h_{u_1} h_{u_2} A_{u_3})\right] \\
\operatorname{rot} \boldsymbol{A} = \nabla \times \boldsymbol{A} &= \frac{1}{h_{u_2} h_{u_3}}\boldsymbol{e}_{u_1}\left[\frac{\partial}{\partial u_2}(h_{u_3} A_{u_3}) - \frac{\partial}{\partial u_3}(h_{u_2} A_{u_2})\right] \\
&\quad + \frac{1}{h_{u_1} h_{u_3}}\boldsymbol{e}_{u_2}\left[\frac{\partial}{\partial u_3}(h_{u_1} A_{u_1}) - \frac{\partial}{\partial u_1}(h_{u_3} A_{u_3})\right] \\
&\quad + \frac{1}{h_{u_1} h_{u_2}}\boldsymbol{e}_{u_3}\left[\frac{\partial}{\partial u_1}(h_{u_2} A_{u_2}) - \frac{\partial}{\partial u_2}(h_{u_1} A_{u_1})\right] \\
\Delta V &= \frac{1}{h_{u_1} h_{u_2} h_{u_3}}\left[\frac{\partial}{\partial u_1}\left(\frac{h_{u_2} h_{u_3}}{h_{u_1}}\frac{\partial V}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_{u_1} h_{u_3}}{h_{u_2}}\frac{\partial V}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_{u_1} h_{u_2}}{h_{u_3}}\frac{\partial V}{\partial u_3}\right)\right]
\end{aligned}
$$

Nützliche Relationen:

$$
\begin{aligned}
\operatorname{div}\operatorname{grad} V &= \nabla \cdot \nabla V = \Delta V \\
\operatorname{rot}\operatorname{grad} V &= \nabla \times \nabla V = 0 \\
\operatorname{div}\operatorname{rot} \boldsymbol{A} &= \nabla \cdot (\nabla \times \boldsymbol{A}) = 0 \\
\nabla(V + W) &= \nabla V + \nabla W \\
\nabla(VW) &= W\nabla V + V\nabla W \\
\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) &= \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A}) + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} \\
\nabla \cdot (\boldsymbol{A} + \boldsymbol{B}) &= \nabla \cdot \boldsymbol{A} + \nabla \cdot \boldsymbol{B} \\
\nabla \times (\boldsymbol{A} + \boldsymbol{B}) &= \nabla \times \boldsymbol{A} + \nabla \times \boldsymbol{B} \\
\nabla \cdot (V\boldsymbol{A}) &= \nabla V \cdot \boldsymbol{A} + V\nabla \cdot \boldsymbol{A} \\
\nabla \times (V\boldsymbol{A}) &= \nabla V \times \boldsymbol{A} + V\nabla \times \boldsymbol{A} \\
\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) &= \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B}) \\
\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) &= \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} \\
\nabla \times (\nabla \times \boldsymbol{A}) &= \nabla(\nabla \cdot \boldsymbol{A}) - \Delta\boldsymbol{A}
\end{aligned}
$$