EUROGRAPHICS 2009 / H.-C. Hege, I. Hotz, and T. Munzner
(Guest Editors)

Volume 28 (2009), Number 3

The Gödel Engine (additional material): Derivation of the special solution to the geodesic equations

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1. Introduction

Our contribution *The Gödel Engine - An interactive approach to visualization in general relativity* uses the analytical solution to the geodesic equations of Gödel's universe to enable interactive renderings including a local illumination model. Section 5.1 of the paper provides compact instructions on solving these equations. Although this section is self-contained, we provide this detailed version of the derivation to simplify the verification of our results. Due to the page limitation and the focus of the conference these calculations have not been included in the paper itself.

All differentiations are with respect to the affine curve parameter λ unless otherwise denoted.

2. Gödel's metric

In general relativity, distances in spacetime are calculated using the line element

$$\mathrm{d}s^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu}(\mathbf{x})\mathrm{d}x^\mu \mathrm{d}x^\nu .$$

This is comparable to the Pythagorean theorem for the distance between two points in Euclidian space. Here, the points on a four-dimensional pseudo-Riemannian manifold are infinitesimally neighboring and the metric tensor $g_{\mu\nu}$, a 4×4 matrix, is introduced. Note that we use index notation $g_{\mu\nu}$ when referring to the entries of the metric tensor $\mathbf{g}$.

In Gödel's universe we obtain a line element [KWSD04]

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1+[r/(2a)]^2} + r^2\left(1-[r/(2a)]^2\right)\mathrm{d}\varphi^2 + dz^2 - \frac{\sqrt{2}r^2c}{a}\mathrm{d}t\mathrm{d}\varphi$$

and therefore a metric tensor

$$\mathbf{g} = \begin{pmatrix} -c^2 & 0 & -\frac{r^2c}{\sqrt{2}a} & 0 \\ 0 & \frac{1}{1+[r/(2a)]^2} & 0 & 0 \\ -\frac{r^2c}{\sqrt{2}a} & 0 & r^2\left(1-[r/(2a)]^2\right) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

This metric tensor has a different algebraic sign compared to [KWSD04]. The underlying physics remain unaffected by this convention.

3. Lagrange formalism

We can formulate the Lagrangian or Lagrange function of motion [Rin01] which reads

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = g_{\mu\nu}u^\mu u^\nu = \kappa c^2. \tag{2}$$

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The equations of motion are second-order ordinary differential equations. They can be derived using the Lagrangian, namely

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^{\mu}} - \frac{\partial\mathcal{L}}{\partial x^{\mu}} = 0. \tag{3}$$

The parameter κ denotes the type of geodesic, we have

$$\begin{aligned} \kappa &= -1 &:& \quad \text{timelike geodesics,} \\ \kappa &= 0 &:& \quad \text{lightlike geodesics,} \\ \kappa &= 1 &:& \quad \text{spacelike geodesics.} \end{aligned}$$

We can find constants of motion if a spacetime has certain symmetries. This enables us to formulate an equivalent set of equations of motion which are of first order in λ. These constants can be found, for example, by applying the Noether theorem

$$\mathcal{L} \neq \mathcal{L}(x^{\mu}) \rightarrow \frac{\partial\mathcal{L}}{\partial\dot{x}^{\mu}} = \text{const.} \tag{4}$$

4. Equations of motion in Gödel's universe

With Eqns. 1 and 2 the Langrangian of Gödel's universe reads

$$\mathcal{L} = -c^2\dot{t}^2 + \frac{\dot{r}^2}{1+[r/(2a)]^2} + r^2\left(1-[r/(2a)]^2\right)\dot{\varphi}^2 + \dot{z}^2 - \frac{\sqrt{2}r^2c}{a}\dot{t}\dot{\varphi} = \kappa c^2 \tag{5}$$

and is independent of t, φ and z. Using Eq. 4 we find three constants of motion

$$k_0 = -c\dot{t} - \frac{r^2}{\sqrt{2}a}\dot{\varphi}, \tag{6a}$$

$$k_2 = r^2\left(1-[r/(2a)]^2\right)\dot{\varphi} - \frac{r^2c}{\sqrt{2}a}\dot{t}, \tag{6b}$$

$$k_3 = \dot{z}. \tag{6c}$$

These expressions are constant for every λ for a particular geodesic. We can therefore use these constants of motion to specify the initial conditions $x^{\mu}(0)$ and $\dot{x}^{\mu}(0)$ of this geodesic:

$$k_0 = -c\dot{t}(0) - \frac{r^2(0)}{\sqrt{2}a}\dot{\varphi}(0), \tag{7a}$$

$$k_2 = r^2(0)\left(1-[r(0)/(2a)]^2\right)\dot{\varphi}(0) - \frac{r^2(0)c}{\sqrt{2}a}\dot{t}(0), \tag{7b}$$

$$k_3 = \dot{z}(0). \tag{7c}$$

We solve Eqns. 6 with respect to $\dot{x}^{\mu}$ and arrive at

$$c\dot{t} = -k_0\frac{1-[r/(2a)]^2}{1+[r/(2a)]^2} - k_2\frac{1}{\sqrt{2}a\left(1+[r/(2a)]^2\right)}, \tag{8a}$$

$$\dot{\varphi} = \frac{k_2 - r^2k_0/(\sqrt{2}a)}{r^2\left(1+[r/(2a)]^2\right)}, \tag{8b}$$

$$\dot{z} = k_3. \tag{8c}$$

These expressions are then inserted into the Lagrangian (Eq. 5) to eliminate $\dot{t}, \dot{\varphi}$ and $\dot{z}$. The Lagrangian now depends only on $r(\lambda)$ and its derivative:

$$\mathcal{L} = \frac{\dot{r}^2 - k_0^2\left(1-[r/(2a)]^2\right) + k_2^2/r^2 - \sqrt{2}k_0k_2/a}{1+[r/(2a)]^2} + k_3^2 = \kappa c^2. \tag{9}$$

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We obtain a coupled first-order ordinary differential equations system using Eqns. 8 and 9:

$$c\dot{t} = -k_0 \frac{1-[r/(2a)]^2}{1+[r/(2a)]^2} - k_2 \frac{1}{\sqrt{2}a\left(1+[r/(2a)]^2\right)}, \tag{10a}$$

$$\dot{r}^2 = (\kappa c^2 - k_3^2)\left(1+[r/(2a)]^2\right) - \frac{k_2^2}{r^2} + \frac{\sqrt{2}k_0 k_2}{a} + k_0^2\left(1-[r/(2a)]^2\right), \tag{10b}$$

$$\dot{\varphi} = \frac{k_2 - r^2 k_0/(\sqrt{2}a)}{r^2\left(1+[r/(2a)]^2\right)}, \tag{10c}$$

$$\dot{z} = k_3. \tag{10d}$$

The equations presented here are formulated for arbitrary initial conditions. We now restrict the solution to special initial conditions. Within the context of the paper we are only interested in geodesics starting at the origin. Isometries are used to obtain indirect access to the general solution of the geodesic equations. For further details see Sec. 5.3.

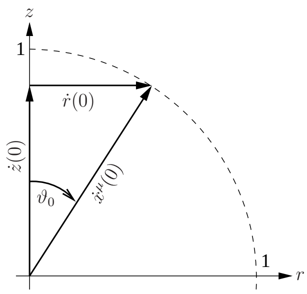
Starting at the origin $r(0) = 0$, Eqns. 7 simplify to

$$k_0 = -c\dot{t}(0), \tag{11a}$$

$$k_2 = 0, \tag{11b}$$

$$k_3 = \dot{z}(0). \tag{11c}$$

These equations show that k_0 can be identified with the negative time direction and we therefore choose $k_0 = \pm 1$. A positive sign denotes geodesics evolving into the past while a negative sign corresponds to geodesics propagating into the future. When we investigate the visual appearance of an object as seen by an observer we need $k_0 = +1$, because we trace the geodesics back into the past starting at the observer's position. If we illuminate an object, we use geodesics propagating into the future. Constant of motion k_3 is the component of the geodesics's starting direction along the z-axis. We restrict $\dot{z}(0) = k_3 \in [-1;1]$ and obtain $\dot{r}(0) = (1-k_3^2)^{1/2}$ for photons leaving the origin $r = 0$.



The vertical starting angle ϑ_0 results to

$$\vartheta_0 = \arctan\left(\frac{\dot{r}}{\dot{z}}\right) = \arccos\left(\frac{\dot{z}}{\dot{r}^2 + \dot{z}^2}\right) = \arccos(k_3)$$

as stated in the paper.

The final differential equations for lightlike geodesics ($\kappa = 0$) and $k_0 = \pm 1$ result from inserting Eqns. 11 in Eqns. 10. In geometrical units ($c = 1$), they can be written as

$$\dot{t} = -k_0 \frac{1-[r/(2a)]^2}{1+[r/(2a)]^2}, \tag{12a}$$

$$\dot{r}^2 = 1 - k_3^2 - (1+k_3^2)[r/(2a)]^2, \tag{12b}$$

$$\dot{\varphi} = \frac{-k_0}{\sqrt{2}a\left(1+[r/(2a)]^2\right)}, \tag{12c}$$

$$\dot{z} = k_3. \tag{12d}$$

Eqns. 12a and 12c are coupled to Eq. 12b. Eq. 12d is trivial. The integration of these equations can be verified using the integration tables of [BSMM07] and elementary trigonometric transformations or by inserting the solution into Eqns. 12.

We will solve Eq. 12b first.

5. Solution to the geodesic equations

Radial coordinate

After separation of variables we obtain

$$\frac{\pm \mathrm{d}r}{\sqrt{4a^2(1-k_3^2)/(1+k_3^2) - r^2}} = \frac{\sqrt{1+k_3^2}}{2a}\,\mathrm{d}\lambda,$$

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where the two signs on the left-hand side result from extracting the root of Eq. 12b and describe a photon leaving from or arriving at the origin, respectively. This equation is integrated, which yields

$$\pm\arcsin\left(\frac{r}{2a}\sqrt{\frac{1+k_3^2}{1-k_3^2}}\right)-r_0=\frac{\sqrt{1+k_3^2}}{2a}(\lambda-\lambda_{0\pm})$$

with two different integration constants $\lambda_{0\pm}$ depending on the branch of the solution. Furthermore we set $r_0=0$ due to our initial conditions. This can be written as

$$r(\lambda)=\pm 2a\sqrt{\frac{1-k_3^2}{1+k_3^2}}\sin\left(\frac{\sqrt{1+k_3^2}}{2a}(\lambda-\lambda_{0\pm})\right).$$

The different branches of the solution are merged to a continuous function for both incoming and outgoing photons and the initial condition $r(0)=0$

$$r(\lambda)=2a\sqrt{\frac{1-k_3^2}{1+k_3^2}}\left|\sin\left(\frac{\sqrt{1+k_3^2}}{2a}\lambda\right)\right|, \tag{13}$$

which is as introduced in the paper.

Time coordinate

After inserting Eq. 13 into 12a we find

$$\dot{t}=-k_0\frac{(1+k_3^2)/(1-k_3^2)-\sin^2\left(\sqrt{1-k_3^2}/(2a)\,\lambda\right)}{(1+k_3^2)/(1-k_3^2)+\sin^2\left(\sqrt{1-k_3^2}/(2a)\,\lambda\right)}$$

which integrates to

$$t(\lambda)=k_0\lambda-2\sqrt{2}ak_0\arctan\left(\frac{\sqrt{2}}{\sqrt{1+k_3^2}}\tan\left(\frac{\sqrt{1+k_3^2}}{2a}\lambda\right)\right)+\pi\left\lfloor\frac{\sqrt{1+k_3^2}}{2a\pi}\lambda+\frac{1}{2}\right\rfloor+t_s.$$

The floor function $\lfloor x\rfloor$ is introduced, because $\arctan(\tan(x))\neq x$ but resembles a discontinuous function. In mathematical terms $\arctan(\tan(x))$ is a piecewise linear function ("sawtooth function"), because $-\pi/2<\arctan(y)<\pi/2\ \forall y\in\mathbb{R}$. Adding an appropriate piecewise constant function results to $\arctan(\tan(x))+\pi\lfloor x/\pi+1/2\rfloor=x$. Hence, the solution is continuously differentiable for all λ. We set the integration constant $t_s=t(0)$ w.l.o.g. to zero.

Angular coordinate

We insert Eq. 13 into 12c, which takes the form

$$\dot{\varphi}=\frac{-k_0(1+k_3^2)/(1-k_3^2)}{\sqrt{2}a\left((1+k_3^2)/(1-k_3^2)+\sin^2\left(\sqrt{1+k_3^2}/(2a)\lambda\right)\right)}$$

and is solved by

$$\varphi(\lambda)=-k_0\arctan\left(\frac{\sqrt{2}}{\sqrt{1+k_3^2}}\tan\left(\frac{\sqrt{1+k_3^2}}{2a}\lambda\right)\right)+\pi\left\lfloor\frac{\sqrt{1+k_3^2}}{2a\pi}\lambda+\frac{1}{2}\right\rfloor-\pi\left\lfloor\frac{\sqrt{1+k_3^2}}{2a\pi}\lambda\right\rfloor+\varphi_s.$$

The function $\lfloor x\rfloor$ guarantees the continuous differentiability of the angular solution. The integration constant $\varphi_s=\varphi(0)$ must be interpreted as the angular starting direction of the geodesic within the $r\varphi$-plane.

z-coordinate

Obviously, Eq. 12d is solved by

$$z(\lambda)=k_3\lambda+z_s,$$

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where we set $z_3 = 0$ due to the initial conditions.

Abbreviations

With the help functions

$$
\begin{aligned}
f_q(\lambda) &= \pi \left\lfloor \frac{\sqrt{1+k_3^2}}{2a\pi}\lambda + q \right\rfloor, \\
g(\lambda) &= \arctan\left(\frac{\sqrt{2}}{\sqrt{1+k_3^2}} \tan\left(\frac{\sqrt{1+k_3^2}}{2a}\lambda \right) \right),
\end{aligned}
$$

the calculations provided here reproduce the analytical solution as presented in the paper.

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