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Periodic orbit quantization of chaotic maps by harmonic inversion

Kirsten Weibert, Jörg Main *, Günter Wunner

Institut für Theoretische Physik 1, Universität Stuttgart, D-70550 Stuttgart, Germany

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Abstract

A method for the semiclassical quantization of chaotic maps is proposed, which is based on harmonic inversion. The power of the technique is demonstrated for the baker's map as a prototype example of a chaotic map.

The harmonic inversion method for signal processing [1,2] has proven to be a powerful tool for the semiclassical quantization of chaotic as well as integrable dynamical systems [3–5]. Starting from Gutzwiller's trace formula for chaotic systems, or the Berry–Tabor formula for integrable systems [6], the harmonic inversion method is able to circumvent the convergence problems of the periodic orbit sums and to directly extract the semiclassical eigenvalues from a relatively small number of periodic orbits. The technique has successfully been applied to a large variety of Hamiltonian systems [4,5]. It has been shown that the method is universal in the sense that it does not depend on any special properties of the dynamical system.

In this Letter we demonstrate that the range of application of the harmonic inversion method extends beyond Hamiltonian systems also to quantum maps. Starting from the analogue of Gutzwiller's trace formula for chaotic maps, we show that the semiclassical eigenvalues of chaotic maps can be determined by a procedure very similar to the one for flows. As an example system we consider the well known baker's map. For this map we can take advantage of the fact that the periodic orbit parameters can be determined analytically.

We briefly review the basics of quantum maps that are relevant to what follows (for a detailed account of quantum maps see, e.g., Ref. [7]). We consider quantum maps, acting on a finite-dimensional Hilbert space of dimension N, which possess a well-defined classical limit for $N \to \infty$. The quantum dynamics is determined by the equation

$$\psi_{n+1} = U\psi_n, \tag{1}$$

where U is a unitary matrix of dimension N, and ψ_n is the N-dimensional discretized wave vector. The eigenvalues u_k of U lie on the unit circle, $u_k = \exp(-i\varphi_k)$. The density of eigenphases φ_k on the unit circle is given by

$$\rho_{\rm qm}(\varphi) = \frac{N}{2\pi} + \frac{1}{\pi}\,{\rm Re}\sum_{n=1}^{\infty} {\rm Tr}\,U^n e^{in\varphi}, \tag{2}$$

which can be rewritten as

$$\rho_{\rm qm}(\varphi) = -\frac{1}{\pi}\,{\rm Im}\,g_{\rm qm}(\varphi) \tag{3}$$

* Corresponding author.
E-mail address: main@theo1.physik.uni-stuttgart.de (J. Main).

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with the response function g_{qm} given by

$$g_{\mathrm{qm}}(\varphi) = g_0(\varphi) - i\sum_{n=1}^{\infty} \mathrm{Tr}\, U^n e^{in\varphi}. \tag{4}$$

In analogy with the periodic orbit theory for flows, a semiclassical approximation to the response function (4) can be obtained in terms of the periodic orbits of the corresponding classical system. (In the context of maps, "periodic orbit" means a sequence of fixed points periodic after n iterations; cyclic shifts of the same sequence correspond to the same periodic orbit.) In the semiclassical approximation for maps, the traces of U^n are related to the periodic orbits of topological length n,

$$\mathrm{Tr}\, U^n \approx \sum_{\mathrm{po}(n)} \frac{n_0}{|\det(M_{\mathrm{po}} - 1)|^{1/2}} e^{i(S_{\mathrm{po}}/\hbar - \mu_{\mathrm{po}}\pi/2)} =: i\mathcal{A}_n, \tag{5}$$

where the sum runs over all periodic orbits of topological length n including multiple traversals of shorter orbits. Here, S_{po} is the action associated with the periodic orbit, μ_{po} is its Maslov index, M_{po} is the monodromy matrix of the orbit, and n_0 is the topological length of the underlying primitive orbit (i.e., the length of the shortest subperiod). The value of the Planck constant is related to the dimension of the Hilbert space via $\hbar = 1/(2\pi N)$. The accuracy of the semiclassical approximation can therefore be expected to improve with increasing N.

The central idea for applying harmonic inversion to the periodic orbit quantization of maps now is to adjust the semiclassical response function

$$g(\varphi) = g_0(\varphi) + \sum_n \mathcal{A}_n e^{in\varphi} \tag{6}$$

to the form of the exact quantum response function (4), expressed in terms of the eigenphases φ_k and their multiplicities m_k,

$$g_{\mathrm{qm}}(\varphi) = \sum_k \frac{m_k}{\varphi - \varphi_k}. \tag{7}$$

It should be pointed out that for all maps by virtue of (4) the semiclassical amplitudes $\mathcal{A}_n$ are independent of the phase φ. In analogy with the harmonic inversion procedures for Hamiltonian flows [3,4], we Fourier transform the oscillating part of the semiclassical response function (6) to obtain the semiclassical signal

$$C(s) = \sum_n \mathcal{A}_n \delta(s - n). \tag{8}$$

The eigenphases φ_k can now be determined by adjusting the semiclassical signal (8) to the form of the corresponding exact quantum signal (the Fourier transform of the exact response function (7))

$$C_{\mathrm{qm}}(s) = -i\sum_k m_k e^{-is\varphi_k} \tag{9}$$

by harmonic inversion. Note that compared to the corresponding procedure for flows, the topological length now plays the role of the scaled action, while the actions S_{po} of the orbits are included in the amplitudes $\mathcal{A}_n$. Therefore, all orbits of the same topological length n contribute to the same signal point. While for flows the semiclassical signal takes on the simple form of a sum over δ functions only if we assume a certain scaling property [4,5], the form of the signal for maps is always the same, as the amplitudes $\mathcal{A}_n$ are for all maps independent of φ.

We will now apply the general procedure discussed above to the example of the baker's map, which has been used as a prototype example for studying the semiclassics of chaotic maps in many investigations in recent years [8–13]. The classical baker's map acts on points (p, q) of the unit square $[0, 1] \times [0, 1]$ according to

$$q' = 2q \bmod 1, \tag{10}$$

$$p' = \big(p + [2q]\big)/2, \tag{11}$$

where $[x]$ denotes the integer part of x. The periodic orbits of the map can be described by a complete binary symbolic code. Each orbit is characterized by a symbol string $\{\epsilon_1, \ldots, \epsilon_n\}$, where $\epsilon_i \in \{0, 1\}$. The action associated with a periodic orbit is given by [9]

$$S_\nu = \frac{\nu\bar{\nu}}{2^n - 1} \bmod 1 \tag{12}$$

with the integers ν and $\bar{\nu}$ defined by

$$\nu = \sum_{k=1}^{n} \epsilon_k 2^{k-1}, \qquad \bar{\nu} = \sum_{k=1}^{n} \epsilon_k 2^{n-k}. \tag{13}$$

Each orbit of length n has stability (i.e., largest eigenvalue of the monodromy matrix) 2^n.

A quantized version of the baker's map which preserves the classical symmetries was derived by Saraceno [8]. For even dimension N of the Hilbert space the quantum map is given by the unitary matrix

$$U(N) = F_N^{-1} \times \begin{pmatrix} F_{N/2} & 0 \\ 0 & F_{N/2} \end{pmatrix} \tag{14}$$

with

$$(F_N)_{nm} = \frac{1}{\sqrt{N}} \exp\left[-2\pi i \left(n - \frac{1}{2}\right)\left(m - \frac{1}{2}\right)\right], \quad n, m = 1, \ldots, N. \tag{15}$$

We will use the the exact quantum eigenvalues for comparison with the semiclassical ones obtained by harmonic inversion of the semiclassical signal (8).

For the baker's map, the semiclassical amplitudes $\mathcal{A}_n$ in signal (8) as defined in Eq. (5) read [11]

$$\mathcal{A}_n = -i \sum_{\mathrm{po}(n)} \frac{2^{n/2} n_0}{2^n - 1} \exp\left(2\pi i N \frac{\nu \bar{\nu}}{2^n - 1}\right) \tag{16}$$

with ν and $\bar{\nu}$ given by Eq. (13). We have performed calculations for dimensions $N = 6$ and $N = 12$. The case $N = 6$ has also been examined in Ref. [11], where the semiclassical eigenvalues were determined by a resummation of the Selberg zeta function of the map. For the construction of the semiclassical signal, we calculated all orbits up to symbol length $n = 20$ for dimension $N = 6$ and up to length $n = 38$ for $N = 12$. Fig. 1 shows the results for the eigenvalues $u_k = \exp(-i\varphi_k)$ obtained by harmonic inversion of the semiclassical signal ($\boxdot$), compared with the exact quantum results ($*$) obtained by diagonalization of the quantum matrix U (see Eq. (14)). For comparison, in the case $N = 6$, we have also plotted the semiclassical results from Ref. [11] ($+$).

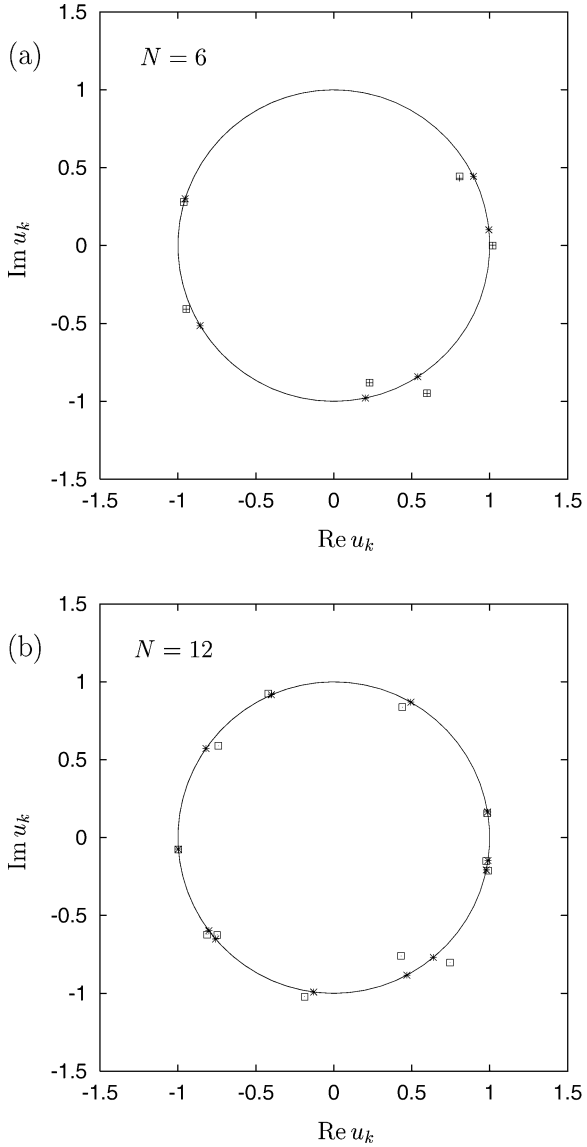

Fig. 1. Exact quantum ($*$) and semiclassical ($\boxdot$) eigenvalues $u_k = \exp(-i\varphi_k)$ of the baker's map obtained by harmonic inversion for dimensions (a) $N = 6$ and (b) $N = 12$. In the case $N = 6$ the semiclassical results from Ref. [11] are also plotted ($+$).

Our semiclassical results are in good agreement with the exact quantum eigenvalues. However, as was also pointed out in Ref. [11], in the case of the baker's map the semiclassical error is relatively large, which is probably due to the discontinuities inherent in this map. In particular, a few of the semiclassical eigenvalues are located away from the unit circle at distances on the order of 10^{-1}. Although the accuracy of the semiclassical approximation should improve in general with increasing N, the doubling of the dimension of the Hilbert space from $N = 6$ to $N = 12$ on average does not yet visibly lead the semiclassical eigenvalues closer to the unit circle.

On the other hand, in the case $N = 6$, the semiclassical eigenvalues obtained by harmonic inversion are in excellent agreement with those from Ref. [11]. This implies that the deviations from the exact quantum eigenvalues are indeed solely due to the semiclassical error, and do not indicate any inaccuracies of the individual methods applied. We add that the semiclassical approximation for the baker's map may even be improved by introducing non-semiclassical correction factors to the amplitudes $\mathcal{A}_n$ [11,12].

As for flows, the harmonic inversion method for maps can also be used, vice versa, to analyze the quantum spectrum in terms of the amplitudes $\mathcal{A}_n$ [14]. This is achieved by adjusting the quantum response function (7) to its semiclassical approximation (6). However, since for maps the amplitudes $\mathcal{A}_n$ contain contributions from all periodic orbits with topological length n, it is not possible to extract information about single periodic orbits from the quantum spectrum. Moreover, since the traces $\operatorname{Tr} U^n$ do not depend on the quantity φ which is to be quantized, the exact expression (4) for maps already fulfills the ansatz of the harmonic inversion procedure as a whole. This is in contrast to scaling Hamiltonian systems, where the higher-order $\hbar$ corrections to the semiclassical approximation depend on the scaling parameter (e.g., for billiard systems, the wave number) and the harmonic inversion of the exact quantum spectrum yields only the zeroth-order $\hbar$ contributions to the response function, with the higher orders acting as a kind of noise [4,5]. As a consequence, the analysis of the quantum spectrum of maps will simply yield the exact quantum values for the traces $\operatorname{Tr} U^n$ rather than their semiclassical approximations $\mathcal{A}_n$ (we have verified this for the baker's map), and no information about the semiclassics or single periodic orbits can be obtained.

In conclusion, we have presented a method for the periodic orbit quantization of chaotic maps, which makes use of harmonic inversion. The procedure works similar to the one for flows, and can in the same way be applied to all chaotic maps, independent of any special properties of the respective system. We have demonstrated the power of the method by successfully applying it to the baker's map. Our results reproduce the exact quantum eigenvalues of the baker's map to within the error of the semiclassical approximation, and are in excellent agreement with those obtained by other semiclassical methods.

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References

[1] M.R. Wall, D. Neuhauser, J. Chem. Phys. 102 (1995) 8011.
[2] V.A. Mandelshtam, H.S. Taylor, J. Chem. Phys. 107 (1997) 6756.
[3] J. Main, V.A. Mandelshtam, H.S. Taylor, Phys. Rev. Lett. 79 (1997) 825.
[4] J. Main, Phys. Rep. 316 (1999) 233.
[5] K. Weibert, J. Main, G. Wunner, Eur. Phys. J. D 12 (2000) 381.
[6] M.C. Gutzwiller, Chaos in Classical and Quantum Mechanics, Springer, New York, 1990.
[7] F. Haake, Quantum Signatures of Chaos, 2nd ed., Springer Series in Synergetics, Vol. 54, Springer, Berlin, 2001.
[8] M. Saraceno, Ann. Phys. 199 (1990) 37.
[9] A.M. Ozorio de Almeida, M. Saraceno, Ann. Phys. 210 (1991) 1.
[10] M. Saraceno, A. Voros, Chaos 2 (1992) 99.
[11] B. Eckhardt, F. Haake, J. Phys. A 27 (1994) 4449.
[12] M. Saraceno, A. Voros, Physica D 79 (1994) 206.
[13] G. Tanner, J. Phys. A 32 (1999) 5071.
[14] J. Main, V.A. Mandelshtam, H.S. Taylor, Phys. Rev. Lett. 78 (1997) 4351.