Bifurcations and exceptional points in a $\mathcal{PT}$-symmetric dipolar Bose-Einstein condensate

Robin Gutöhrlein, Holger Cartarius, Jörg Main and Günter Wunner

1. Institut für Theoretische Physik, Universität Stuttgart, 70550 Stuttgart, Germany

E-mail: `Robin.Gutoehrlein@itp1.uni-stuttgart.de`

Abstract. We investigate the bifurcation structure of stationary states in a dipolar Bose-Einstein condensate located in an external $\mathcal{PT}$-symmetric potential. The imaginary part of this external potential allows for the effective description of in- and out-coupling of particles. To unveil the complete bifurcation structure and the properties of the exceptional points we perform an analytical continuation of the Gross-Pitaevskii equation, which is used to describe the system. We use an elegant and numerically efficient method for the analytical continuation of the Gross-Pitaevskii equation with dipolar interactions containing bicomplex numbers. The Bose-Einstein condensate with dipole interaction shows a much richer bifurcation scenario than a condensate without long-range interactions. The inclusion of analytically continued states can also explain property changes of bifurcation points which were hidden before, and allows for the examination of the properties of the exceptional points associated with the branch points. With the new analytically continued states we are able to prove the existence of an exceptional point of fifth order.

1. Introduction

Recently $\mathcal{PT}$-symmetric systems have gained much attention. Such systems do feature a special class of non-Hermitian Hamiltonians which exhibit special properties such as a real eigenvalue spectrum [1]. An operator is considered to be $\mathcal{PT}$-symmetric if it commutes with the $\mathcal{PT}$-operator,

$$[\mathcal{PT}, H] = 0, \tag{1}$$

where $\mathcal{P}$ is the parity operator ($\hat{x} \to -\hat{x}$ and $\hat{p} \to -\hat{p}$), and $\mathcal{T}$ is the time-reversal operator which changes $\hat{p} \to -\hat{p}$ and $\mathrm{i} \to -\mathrm{i}$. For a system of which the Hamiltonian can be written as

$$H = -\Delta + V(x), \tag{2}$$

$\mathcal{PT}$-symmetry implies the condition

$$V(x) = V^*(-x) \tag{3}$$

for the (complex) potential. In such a system the imaginary part of the potential represents the in- and out-coupling of particles into or from an external reservoir.

Quantum systems fulfilling this property have been studied in, e.g., [1–4]. However, the concept of $\mathcal{PT}$-symmetry is not restricted to quantum mechanics. The first experimental realization of $\mathcal{PT}$-symmetric systems was indeed achieved in optical wave guides where the effects of $\mathcal{PT}$-symmetry and $\mathcal{PT}$-symmetry breaking were observed [5]. These first breakthroughs have increased the research effort put into this field [6–9]. $\mathcal{PT}$-symmetric systems have also been studied in microwave cavities [10], electronic devices [11, 12], and in optical [13–22] systems. Also in quantum mechanics the stationary Schrödinger equation was solved for scattering solutions [3] and bound states [4]. The characteristic $\mathcal{PT}$-symmetric properties are still found when a many-particle description is used [23]. In [24] a $\mathcal{PT}$-symmetric system was embedded as a subsystem into an hermitian system, showing that the subsystem retained its $\mathcal{PT}$-symmetric properties.

In the first realizations of Bose-Einstein condensates [25–27] atoms without long-range interactions were used. Here the dominating interactions can be described by s-wave scattering. However, when other atoms with a large dipolar moment were investigated [28–31] new effects emerged. Depending on the strength of the dipole-dipole interaction a novel behaviour can be introduced into the condensates, such as an anisotropic collapse of ^{52}Cr atoms [32].

Bose-Einstein condensates described by the Gross-Pitaevskii equation can be placed in a $\mathcal{PT}$-symmetric potential. While new effects of condensates with long-range dipole-dipole interactions have been found [33–35] when placed in a double-well trap, completely new structures arise if additional gain and loss terms are introduced.

Different processes for the realisation of loss and gain in a condensate were proposed. For the loss of particles the use of a focused electron beam [36] at the loss site was examined. The gain might be realized by feeding the condensate from a second condensate [37] in a Raman super-radiance-like pumping [38, 39]. Also the coupling to additional potential wells with a higher or lower potential base could be used [40].

Since the $\mathcal{PT}$-symmetry of the Gross-Pitaevskii equation depends on the $\mathcal{PT}$-symmetry of the wave function, effects which change the geometry of the wave function can lead to additional phenomena. Dipolar Bose-Einstein condensates exhibit such effects, e.g. structured ground states have been found [41]. Therefore one would expect that the combination of dipolar Bose-Einstein condensates with a $\mathcal{PT}$-symmetric trap will lead to a new behaviour. In [42] a Bose-Einstein condensate with long-range dipole-dipole interaction in a $\mathcal{PT}$-symmetric double-well potential was examined. This condensate shows a richer, much more elaborate bifurcation scenario with more states involved than in the case of a condensate with only short-range interactions. Some of these bifurcations include up to five states, and therefore allow for the possibile existence of higher-order exceptional points, that is points in the parameter space where the eigenvalues and eigenfunctions coalesce [43–49].

An analytical continuation of the Gross-Pitaevskii equation provides the mathematical tool to examine bifurcations and exceptional points in detail. An

encircling of exceptional points in complex parameter space can reveal, through the exchange behaviour of the participating states, the order of the exceptional point [50]. Also additional states and bifurcations which only exist in the analytically continued space are revealed. We can apply this method to the system investigated in [42] where bifurcations with up to five states have been observed.

In this paper we will examine this bifurcation scenario in much more detail using an analytically continued Gross-Pitaevskii equation, therefore allowing us to find additional (mathematical) states which compose the bifurcations. This mathematical tool also allows us to expose the exchange behaviour of the exceptional points associated with the bifurcations. In section 2 we will first give a short recapitulation of the Gross-Pitaevskii equation and the ansatz of the wave function. In section 3 we will give an introduction to the analytical continuation with bicomplex numbers and the representation in an idempotent basis. Finally in section 4 we will present new features in the bifurcation scenario and examine the properties of the associated exceptional points.

2. Gross-Pitaevskii equation and the time-dependent variational principle

In this section we recapitulate the theoretical description of a Bose-Einstein condensate using an ansatz of coupled Gaussians for the wave function. We use the following parameterization of the Gross-Pitaevskii equation

$$\left[-\Delta + c_{\text{sc}} \left|\psi(\mathbf{x}, t)\right|^2 + V_{\text{d}} + V_{\text{ext}}\right] \psi(\mathbf{x}, t) = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\psi(\mathbf{x}, t) \tag{4}$$

with the scattering length c_{sc}. The long-range dipole interaction is described by the dipole potential

$$V_{\text{d}} = c_{\text{d}} \int \mathrm{d}^3 x' \frac{1 - 3\cos^2\theta}{\left|\mathbf{x} - \mathbf{x}'\right|^3} \left|\psi(\mathbf{x}', t)\right|^2 \tag{5}$$

with the dipole strength $c_{\text{d}} = 1$ and the angle θ between $\mathbf{x} - \mathbf{x}'$ and the dipole alignment. A more complete overview of the theoretical description and experimental realisation can be found in [51]. We describe the external double-well potential by two Gaussian beams [42]

$$V_{\text{ext}} = \sum_{i=1}^{2} c_{\text{g},i} \exp\left(-(\mathbf{x}^T \mathbf{A}_{\text{g},i}\mathbf{x} + \mathbf{x}^T \mathbf{q}_{\text{g},i} + \gamma_{\text{g},i})\right), \tag{6}$$

where $c_{\text{g},i}$ defines the amplitude, $\mathbf{A}$, $\mathbf{q}$ and γ define the shape and location of the Gaussian beams. The two wells are located at $\mathbf{q} = (\pm x_{\text{pos}}, 0, 0)$ symmetrically with respect to the origin and with the amplitude

$$c_{\text{g},1} = V_0 + \mathrm{i}\Gamma \text{ and } c_{\text{g},2} = V_0 - \mathrm{i}\Gamma, \tag{7}$$

and therefore fullfilling the requirement (3) for a $\mathcal{PT}$-symmetric potential. The parameter V_0 represents the strength of the real potential, and Γ determines the strength of the in- and outcoupling of particles from an external reservoir. We are especially interested in how a change of the in- and outcoupling strength Γ effects the bifurcation behaviour.

To obtain the equations of motion we apply the time-dependent variational principle by McLachlan [52] to (4). To fullfil the GPE as best as possible the difference betweeen the two sides of the equation

$$I = ||\mathrm{i}\psi - H\underbrace{\frac{\partial}{\partial t}\psi}_{=:\phi}||^2 \tag{8}$$

is minimized. For a given time t the wave function ψ is pinned and the time-derivative ϕ is varied. Hence, if I is a minimum for small $\delta\phi$ the change of δI must vanish. We make an ansatz of coupled Gaussians for the wave function

$$\psi(\mathbf{x}, t) = \sum_{k=1}^{N} g^k = \sum_{k=1}^{N} \exp\left(-\left(\mathbf{x}^T\mathbf{A}_k\mathbf{x} + \mathbf{p}_k^T\mathbf{x} + \gamma_k\right)\right) \tag{9}$$

where the complex symmetric matrix $\mathbf{A}$ defines the shape of the Gaussian, the complex momentum vector is given by $\mathbf{p}$ and the complex number γ defines the amplitude and phase of the Gaussian. This ansatz does not allow for arbitrary matrices $\mathbf{A}$, e.g. matrices which contain a negative gaussian widths are not normalizable and are not accepted as solutions.

Because our trap exhibits a strong confinement in the y direction [42] we can introduce further restrictions to the ansatz of our wave-function. We do not consider all rotational and translational degrees of freedom in the direction of the confinement $(A_{xy} = A_{yz} = p_y = 0)$. With the help of the time-dependent variational principle one can obtain the time derivatives of the Gaussian parameters $\dot{\mathbf{A}}_k$, $\dot{\mathbf{p}}_k$ and $\dot{\gamma}_k$ [42]. A numerical root search can be performed to obtain the stationary states. It was shown [53] that the variational principle, while being numerically cheap, provides results with extremely high quality. For all calculations in this paper, an ansatz of $N = 2$ coupled Gaussians was chosen, i.e. the wave function is approximated in each well by one Gaussian function (but they are not restricted to a specific location).

3. Analytical continuation

The non-analyticity of the Gross-Pitaevskii equation (4), due to the square modulus of the wave-function, prevents us from examining the exchange behaviour of exceptional points. Also since the equation is not analytical the number of states can change from one side of the bifurcation to the other [42]. It has been shown for simpler systems without long-range interactions that with an analytical continuation [50, 54] the whole bifurcation scenario of the equation is revealed and the complete relation between the different stationary states can be observed. But since the Gross-Piaevskii equation is non-analytical and already in the complex domain, the analytical continuation has to go beyond the complex domain. One way to perform the analytical continuation is to separate all complex equations and complex parameters into twice the number of real equations with twice the number of real parameters. Then these equations can be analytically continued. An equivalent alternative is the introduction of bicomplex

numbers. In this approach the complex numbers $\mathbb{C}$ are replaced by bicomplex numbers $\mathbb{BC}$ [55]. Both methods are mathematically completely equivalent, however, the second approach is much more elegant since it provides huge numerical benefits as we will see in the following discussion. Note that there are different hypercomplex numbers. But for the description we have in mind only bicomplex numbers have the correct properties. Furthermore this choice ensures that while extending the original model, all its properties and states are contained and no imformation is lost. In other scenarios different hypercomplex numbers are used and appropriate, e.g. quaternions and co-quaternions are discussed in [56, 57].

3.1. Bicomplex numbers

We first recapitulate some basic properties of bicomplex numbers [55]. In contrast to complex numbers bicomplex numbers have three imaginary units with the following properties

$$k = ij , \quad i^2 = j^2 = -1 , \quad k^2 = 1. \tag{10}$$

To analytically continue a complex equation we split the equation into its real and imaginary part

$$z = \underbrace{\underbrace{z_{\mathrm{r}}}_{\in\mathbb{R}} + i \underbrace{z_{\mathrm{i}}}_{\in\mathbb{R}}}_{\in\mathbb{C}} . \tag{11}$$

These real and imaginary parts can be extended to complex numbers with the imaginary unit j. Then z becomes a bicomplex number

$$z = \underbrace{(\underbrace{z_0}_{\in\mathbb{R}} + j \underbrace{z_1}_{\in\mathbb{R}})}_{\in\mathbb{C}_{\mathrm{j}}} + i \underbrace{(\underbrace{z_2}_{\in\mathbb{R}} + j \underbrace{z_3}_{\in\mathbb{R}})}_{\in\mathbb{C}_{\mathrm{j}}}}_{\in\mathbb{BC}} = z_0 + jz_1 + iz_2 + kz_3. \tag{12}$$

The complex numbers $\mathbb{C}_{\mathrm{j}}$ with the imaginary unit j are isomorphous to the complex numbers $\mathbb{C}_{\mathrm{i}}$ with the imaginary unit i.

3.2. Representation of bicomplex numbers in the idempotent basis

There exists a representation of bicomplex numbers in an idempotent basis. Let us first consider the two idempotent elements

$$e^{\oplus} = \frac{1 + k}{2} \text{ and } e^{\ominus} = \frac{1 - k}{2}, \tag{13}$$

which fullfil the relations

$$e^{\oplus 2} = e^{\oplus}, \ e^{\ominus 2} = e^{\ominus} \text{ and } e^{\oplus}e^{\ominus} = 0. \tag{14}$$

These properties allow us to decompose every bicomplex number of the form (12) into

$$\begin{aligned} z &= ((z_0 + z_3) + (z_2 - z_1)i) \, e^{\oplus} + ((z_0 - z_3) + (z_2 + z_1)i) \, e^{\ominus} \\ &= z^{\oplus}e^{\oplus} + z^{\ominus}e^{\ominus}. \end{aligned} \tag{15}$$

The coefficients $z^\oplus$ and $z^\ominus$ can be chosen to be either elements of $\mathbb{C}_i$ or $\mathbb{C}_j$. In our calculations we choose the $\mathbb{C}_i$ representation. We can now consider the implication the idempotent properties have on the basic arithmetic operations. One finds that [55]

$$a \pm b = (a^\oplus \pm b^\oplus)e^\oplus + (a^\ominus \pm b^\ominus)e^\ominus, \tag{16a}$$

$$a \cdot b = (a^\oplus \cdot b^\oplus)e^\oplus + (a^\ominus \cdot b^\ominus)e^\ominus, \tag{16b}$$

$$a/b = (a^\oplus/b^\oplus)e^\oplus + (a^\ominus/b^\ominus)e^\ominus, \tag{16c}$$

viz. the operations can be performed independently for the plus and minus components in the complex subspaces. These properties allow us to use a very efficient numerical implementation (which uses already existing algorithms). In particular for terms which contain the dipolar potential a highly optimized algorithm for complex numbers is available. If the method without bicomplex numbers would have been used other integrals had to be solved, for which no algorithms exist. We will see in the next section that special care must be taken if a calculation involves the complex conjugate.

3.3. Complex conjugation

There exist different kinds of complex conjugations for bicomplex numbers. In [58] possible conjugations and new $\mathcal{PT}$-symmetries are discussed. However, for our application only one is relevant, since our goal is that the extended model contains all states and properties of the original model. In order to determine which one must be used we first consider the complex conjugation which occurs in (4) without the bicomplex continuation. We start with the complex conjugation of the real and imaginary part of the complex number

$$z^* = z_r - i z_i. \tag{17}$$

Now the analytical continuation is applied by replacing the real and imaginary parts with complex numbers with the imaginary unit j

$$z^* = (z_0 + j z_1) - i(z_2 + j z_3) = z_0 + j z_1 - i z_2 - k z_3. \tag{18}$$

This complex conjugation changes the sign of all i and k components. Using this complex conjungation we can write the modulus squared of the wave function in (4) as

$$|\psi|^2 = \langle \psi | \psi \rangle_{\mathbb{BC}} = \psi^* \psi. \tag{19}$$

This complex conjugation also must be used in inner products. Note that in the representation of the idempotent basis the complex conjugate reads

$$z^* = \overline{z^\ominus} e^\oplus + \overline{z^\oplus} e^\ominus, \tag{20}$$

where the bar denotes the complex conjugation with $z^\oplus, z^\ominus \in \mathbb{C}_i$.

3.4. Decomposition of the GPE using the idempotent basis

With the properties shown so far we can examine the analytical continuation of the Gross-Pitaevskii equation. A linear set of equations, such as the one obtained by the

time-dependent variational principle [59], can be decomposed into two linear systems of equations using the idempotent basis

$$\mathbf{K}^{\oplus}\mathbf{v}^{\oplus} = \mathbf{r}^{\oplus}, \tag{21a}$$

$$\mathbf{K}^{\ominus}\mathbf{v}^{\ominus} = \mathbf{r}^{\ominus}. \tag{21b}$$

The vectors $\mathbf{r}^{\oplus}$, $\mathbf{r}^{\ominus}$ and the matrices $\mathbf{K}^{\oplus}$, $\mathbf{K}^{\ominus}$ contain both plus and minus coefficients of the Gaussian parameters. The elements of the bicomplex matrices $\mathbf{K}$ and vectors $\mathbf{r}$ are of the form [59]

$$\langle g_l | \alpha^n \beta^m V | g_k \rangle_{\mathbb{BC}} \tag{22}$$

with $\alpha, \beta = x, y, z$ and $n, m \in \mathbb{N}$, and the Gaussians g_l, g_k. If the bicomplex conjugate described in (20) is used, one can decompose the expression into

$$\langle g_l | \alpha^n \beta^m | g_k \rangle_{\mathbb{BC}} = \langle g_l^{\ominus} | \alpha^n \beta^m V^{\oplus} | g_k^{\oplus} \rangle \mathrm{e}^{\oplus} + \langle g_l^{\oplus} | \alpha^n \beta^m V^{\ominus} | g_k^{\ominus} \rangle \mathrm{e}^{\ominus}. \tag{23}$$

This representation allows us to use an existing algorithm for complex numbers to calculate the necessary integrals. The algorithm just must be called twice for the appropriate parameter sets. For elliptic integrals the Carlson algorithm [60] can be used. However, the largest numerical improvement is achieved by using the highly optimized algorithm presented in [59] for the numerical integration of the dipolar term.

Finally we examine how the control parameters, which were real numbers before the analytical continuation was applied, can be represented in the idempotent basis. Originally the scattering length c_{sc} was real. After the complex continuation it assumes the form

$$c_{\mathrm{sc}} = c_{\mathrm{sc}}^0 + \mathrm{j}c_{\mathrm{sc}}^1, \tag{24}$$

with $c_{\mathrm{sc}}^0, c_{\mathrm{sc}}^1 \in \mathbb{R}$. We can now represent this parameter in the idempotent basis

$$c_{\mathrm{sc}} = c_{\mathrm{sc}}^0 + \mathrm{j}c_{\mathrm{sc}}^1 = \underbrace{\left(c_{\mathrm{sc}}^0 - \mathrm{i}c_{\mathrm{sc}}^1\right)}_{c_{\mathrm{sc}}^{\oplus}} \mathrm{e}^{\oplus} + \underbrace{\left(c_{\mathrm{sc}}^0 + \mathrm{i}c_{\mathrm{sc}}^1\right)}_{c_{\mathrm{sc}}^{\ominus}} \mathrm{e}^{\ominus}. \tag{25}$$

Since there are only two idependent real parameters the relation $\overline{c_{\mathrm{sc}}^{\oplus}} = c_{\mathrm{sc}}^{\ominus}$ is fullfilled. The same also applies for the in- and out-coupling parameter Γ.

Another special parameter type exists. If the bicomplex number z only contains a real part and an imaginary part for the complex unit i, but not for j or k, the number can be decomposed in the idempotent basis in such a way that the two coefficients are equal, i.e. $z^{\oplus} = z^{\ominus}$.

4. Results

4.1. Bifurcation scenario of a BEC without long-range interaction

To compare the bifurcation scenario between a BEC with and without dipolar interaction, a short recapitulation of the bifurcation scenario of a BEC without long-range interactions is given in this section. This scenario has been discussed in detail in
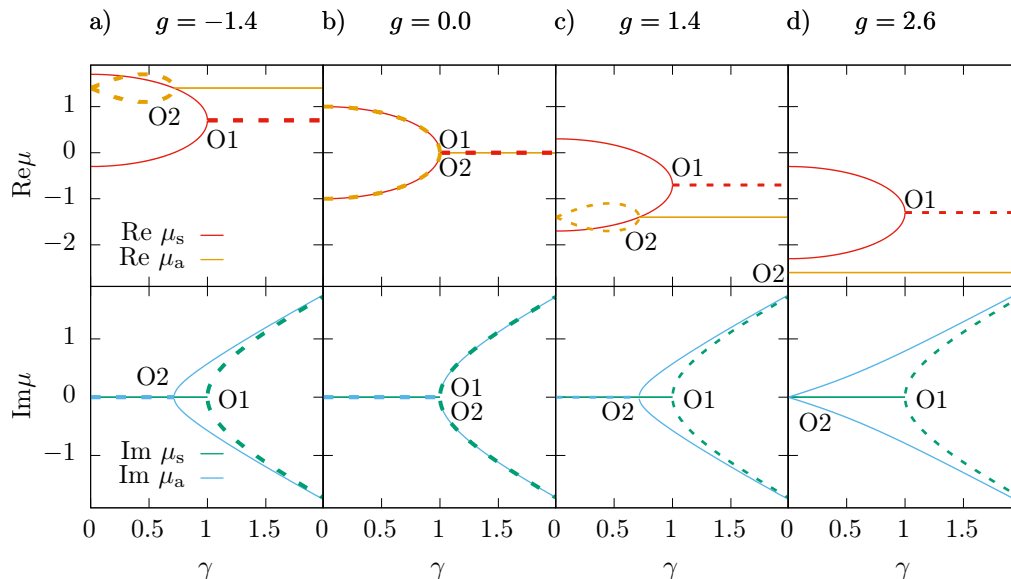
Figure 1. Bifurcation scenarios of a Bose-Einstein condensate without long-range interaction. The underlying equation (26) was analytically continued using bicomplex numbers. Solutions which exist in the complex plane are represented by solid lines. Solutions which require the analytical continuation are shown as dashed lines. States which are $\mathcal{PT}$-symmetric are denoted by the subscript s, while states which break this symmetry are denoted by a.

previous publications [61–64]. The bifurcation scenario of such a $\mathcal{PT}$-symmetric double well system can be described by the matrix equation

$$\begin{pmatrix} -g|\psi_1|^2 - \mathrm{i}\gamma & v \\ v & -g|\psi_2|^2 + \mathrm{i}\gamma \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \mu \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{26}$$

with the coupling strength v, the nonlinearity g, and the in- and outcoupling strength γ.

In the linear case $g = 0$ there exist two parameter ranges for the in- and out-coupling strength γ. In the regime below a critical parameter γ_c there are two $\mathcal{PT}$-symmetric solutions, which merge in a tangent bifurcation (O1 in figure 1(b)) at the critical parameter γ_c. For values greater than this critical strength, there exist two $\mathcal{PT}$-broken solutions with complex chemical potentials. The tangent bifurcation in this scenario is a second-order exceptional point. One can observe the pitchfork bifurcation (O2 in figure 1) which occurs on the upper $\mathcal{PT}$-symmetric branch for values of $-2 < g < 0$ (see figure 1(a)) and switches to the lower $\mathcal{PT}$-symmetric branch for values of $0 < g < 2$ (see figure 1(c)). This bifurcation vanishes for values $g < -2$ or $g > 2$ (see figure 1(d)).

We will see that a bifurcation similar to bifurcation O1 appears in a dipolar Bose-Einstein condensate. The behaviour of the bifurcation O2, which also occurs, will be altered in certain parameter regions. The change in behaviour is also dependent on new emerging bifurcations.
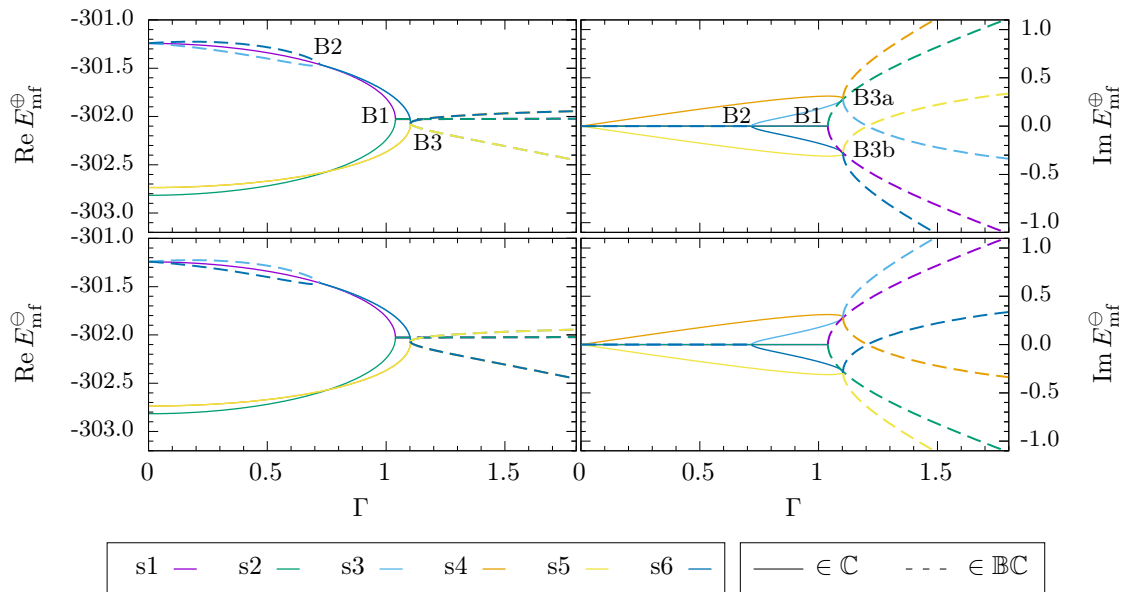
Figure 2. Real and imaginary parts of the plus and minus component of the bicomplex mean-field energy in a dipolar Bose-Einstein condensate with scattering length $c_{sc} = -0.9$. The mean-field energy is composed as $E_{mf} = E_{mf}{}^{\oplus}e^{\oplus} + E_{mf}{}^{\ominus}e^{\ominus}$. States which exist without the analytical continuation into bicomplex values must obey $E_{mf}{}^{\oplus} = E_{mf}{}^{\ominus}$ and are shown as solid lines. States which exist only in the bicomplex domain are shown as dashed lines. The four bifurcations which occur between the states are marked with B1 to B4. As discussed in [42] states s1 and s2 obey $\mathcal{PT}$-symmetry from $\Gamma = 0$ up to the bifurcation B1. Beyond this point the states are $\mathcal{PT}$-broken.

4.2. Results for a BEC with long-range dipole interactions

The calculations for the analytically continued GPE (4) reveals the complete bifurcation structure of the system. In addition to the known stationary states which exist in the complex domain new states which are truly bicomplex are found. With these new states all branches of the bifurcations are present. In addition it is possible to evaluate the exchange behaviour of the exceptional points.

In figure 2 eigenvalues for states with $c_{sc} = -0.9$ are shown. The states which are purely complex (solid lines) are already known from [42]. Since these states are complex the relation

$$E_{mf}{}^{\oplus} = E_{mf}{}^{\ominus} \tag{27}$$

holds true. The real and imaginary parts of the mean-field energy are

$$\mathrm{Re}\, E_{mf} = \frac{1}{2}\left(E_{mf}{}^{\oplus} + E_{mf}{}^{\ominus}\right) \quad \text{and} \quad \mathrm{Im}\, E_{mf} = \frac{i}{2}\left(E_{mf}{}^{\oplus} - E_{mf}{}^{\ominus}\right). \tag{28}$$

The imaginary part of the mean-field energy exists due to the loss or gain of particles in the system. For the states present in the original model the real and imaginary parts of the energy are real numbers. For new analytically continued states these numbers are now complex. The relation (27) no longer holds true for states living only in the bicomplex space (dashed lines). Since all branches of the bifurcations are now present,
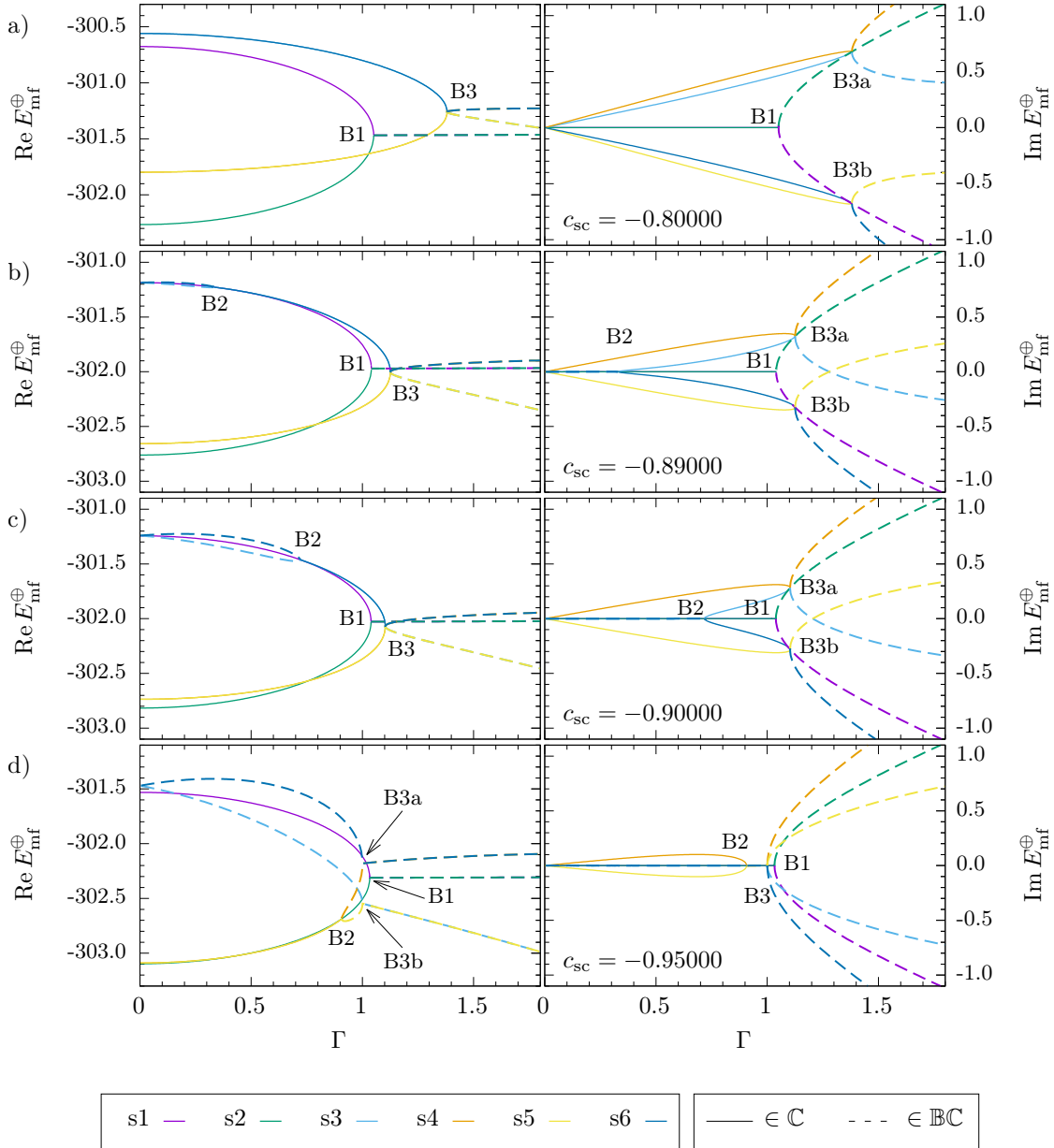
Figure 3. Mean-field energies $E_{\mathrm{mf}}^{\oplus}$ for different values of c_{sc}. The different bifurcation points are denoted by B1, B2, B3a and B3b. When the bifurcations B3a and B3b coincide they are marked by B3. The bifurcation B2 does not exist in (a).

there exist six states for the whole parameter range of Γ. The chemical potential exhibits the same qualitative behaviour as the mean-field energy. Since the qualitative behaviour is the same in the plus and minus components only the plus component of the mean-field energy is discussed in the following.

In figure 3 the real and imaginary parts of the plus component of the mean-field energy are shown for different values of the scattering strength. For small values of Γ the states s1 and s2 show the typical behaviour also known from BECs without long-range
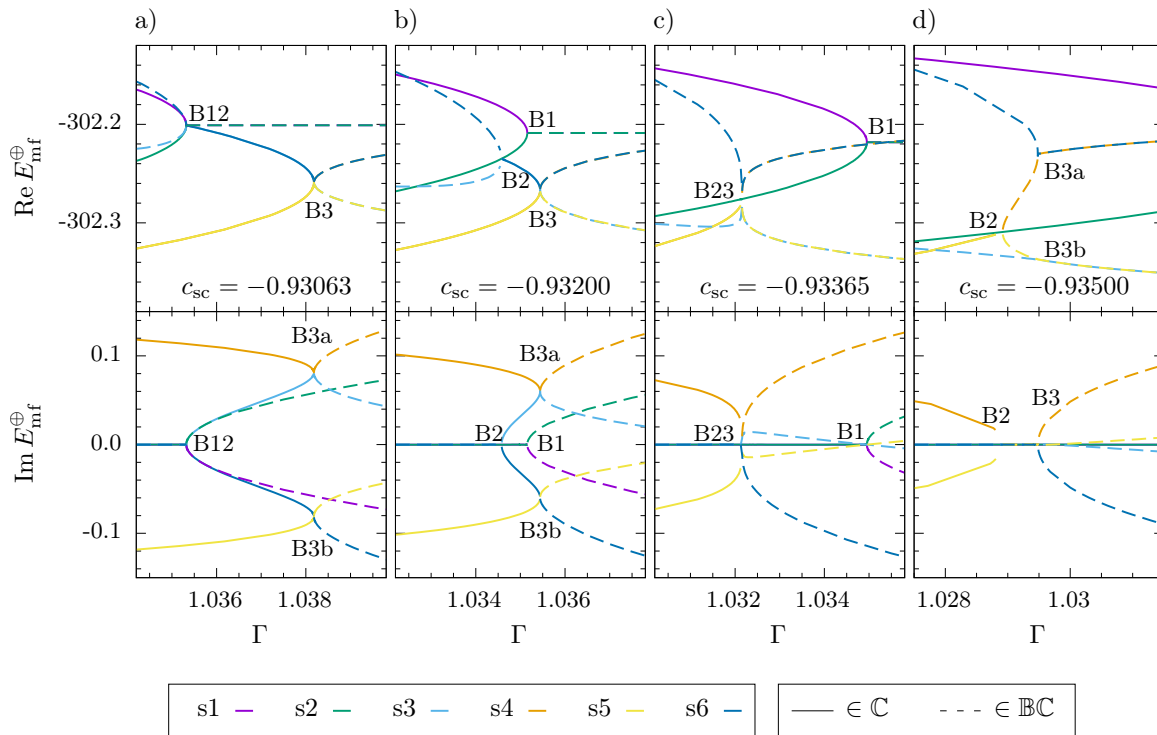
Figure 4. For a scattering length of $c_{\mathrm{sc,crit,1}} = -0.93063$ the bifurcations and EP2s B1 and B2 merge. This can be seen in subfigure a) and the resulting bifurcation is marked as B12. In subfigure c) the EPs and bifurcations B2, B3a and B3b merge for $c_{\mathrm{sc,crit,2}} = -0.93365$ into a higher order exceptional point which is denoted by B23. Subfigure b) shows the bifurcation scenario for an intermediate scattering length of $c_{\mathrm{sc}} = -0.932$.

interactions. They merge in a tangent bifurcation B1. Up to this point the states also exist without the analytical continuation. For larger Γ they become bicomplex. This qualitative behaviour is independent of the scattering length, only the critical point is slightly shifted.

The bifurcation B2 undergoes multiple behaviour changes for different scattering lengths. Also this bifurcation does not always exist (see figure 3(a)). For smaller scattering lengths the bifurcation appears at $\Gamma = 0$. This is a pitchfork bifurcation between the states s1, s3, and s6. The bifurcation moves along the state s1 to larger Γ as the scattering length is decreased (see figure 3(b) and (c)). At a critical scattering length $c_{\mathrm{sc,crit,1}} = -0.93063$ the bifurcation merges with the bifurcation B1 (see figure 4(a)). For even smaller scattering lengths the bifurcation moves back to $\Gamma = 0$ along the state s2. The pitchfork bifurcation is now formed between the states s2, s3, and s6 (see figure 4(b)). However, before the bifurcation reaches $\Gamma = 0$ another critical scattering length is reached. At $c_{\mathrm{sc,crit,2}} = -0.93365$ the bifurcation B2 merges with the bifurcations B3a and B3b (see figure 4(c)). At this point the behaviour of the bifurcation is altered. For larger scattering lengths at smaller Γ the participating states exist only for the bicomplex equations (see figure 4(b)), but for larger Γ the states exist in the complex
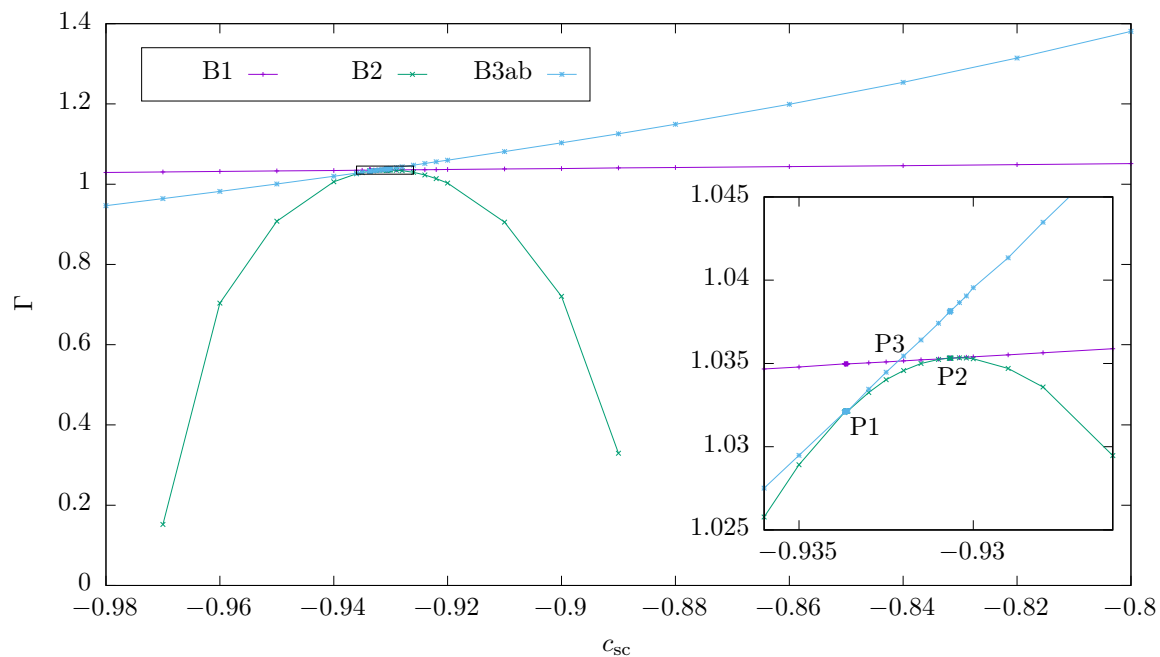
Figure 5. Positions of the bifurcations and exceptional points in the Γ-c_{sc}-parameter space. In the inset the merger of multiple bifurcations can be observed. Note that at the point P3 no merger occurs. There merely exist two bifurcations between different states at the same point in parameter space.

Gross-Pitaevskii equation (see figure 4(d)) without the bicomplex extension. If the scattering length is smaller than $c_{sc,crit,2}$ this behaviour is mirrored with respect to the Γ-axis. Until the bifurcation vanishes for smaller scattering lengths at $\Gamma = 0$ it is composed of the states s2, s4, and s5. If the bifurcation B2 is compared with the bifurcation O2 of the Bose-Einstein condensate without long-range interactions (see figure 1), the first change (the merger of B1 and B2) can also be observed. However, the second change in behaviour is a new effect since the bifurcations B3a and B3b do not exist without long-range interactions.

We have seen that the merger of the bifurcations B2, B3a, and B3b changes the behaviour of bifurcation B2. The properties of the tangent bifurcations B3a and B3b are altered. For scattering lengths greater than $c_{sc,crit,2}$ the bifurcations B3a and B3b are bicomplex for larger Γ (i.e., the states have components of the imaginary units j and k). However, for Γ below the critical value only the real component and the component with complex unit i is nonzero (see figure 4(b)). For smaller values of the scattering lengths the states in both Γ regions are bicomplex, i.e., they only exist as solutions of the analytically continued Gross-Pitaevskii equation (see figure 4(d)).

We have found that the critical scattering length parameters, which divide the parameter region with different behaviours, are related with the merger of multiple bifurcations. In figure 5 the parameter pairs of the scattering length c_{sc} and Γ are shown at which the different bifurcations occur. One observes three points at which the parameters of two different bifurcations become identical. Point P3 is not special, the

states which are involved in the two bifurcations (B1 und B3) have different eigenvalues and wave functions. Therefore just two independent bifurcations occur at the same parameter pair. By contrast at P2 the bifurcations B1 and B2 are joined into one bifurcation. The bifurcation scenario is shown in figure 4(a). Another bifurcation merger appears for the parameters at point P1. The resulting bifurcation, which consits of B2, B3a, and B3b is shown in figure 4(c). These merger points are also of special interest because they have the prerequesites necessary that exceptional points of higher order can appear.

4.3. Exchange behaviour of the states around the exceptional points

We now examine which signatures of exceptional points can be observed. In [47, 50] it was discussed that a complex encircling of a higher order exceptional point does not have to exhibit an exchange of all states which are involved in the exceptional point. Using different parameters to encircle the EP can show different exchange behaviour.

The second-order exceptional points of the tangent bifurcations B1, B3a, and B3b show the expected square root behaviour by exchanging each state with the other (see figure 6(a,c,d)) when the point is encircled in the complex parameter space. On the other hand the third-order exceptional point at the bifurcation B2 only shows the exchange of two states. For the encircling in the complex Γ a cubic root exchange behaviour cannot be observed (see figure 6(b)).

In figure 7(a) we show the exchange behaviour of the states involved in P2 where the bifurcations B1 and B2 merge. When encircling the exceptional point in a complex Γ-parameter plane an exchange within pairs of states is found, however, these two exchanges are separated and no exchange between all four states can be observed. Therefore it is unclear whether these are two second-order exceptional points or one fourth-order exceptional point. Also by encircling the critical point in the complex scattering length plane does not change the qualitative behaviour, again only two states exchange. To prove that this must be an EP4, one must search for further complex perturbation parameters. However, since for an exceptional point of order n, all n eigenvalues and eigenstates must coalesce [43] we examine the wave functions of the participating states and they all coalesce at the critical point (which means that for the ansatz of coupled Gaussians all Gaussian parameters must be the same, which is indeed the case).

The same examination can be performed for the merger of the bifurcations B2, B3a, and B3b (point P1). At this critical point five eigenvalues coalesce. A circle in the complex Γ plane reveals the signature of four exchanging states (see figure 7(b)). Again the circle can be repeated in the complex scattering plane (figure 7(c)) resulting in the same exchange behaviour. In this case the question arises whether this is an EP4 or an EP5. The exchange behaviour proves that the order of the exceptional point must be at least four. All wave functions of the participating states coalesce at the critical point, however, to finally decide whether this is an EP5 further perturbation parameter must
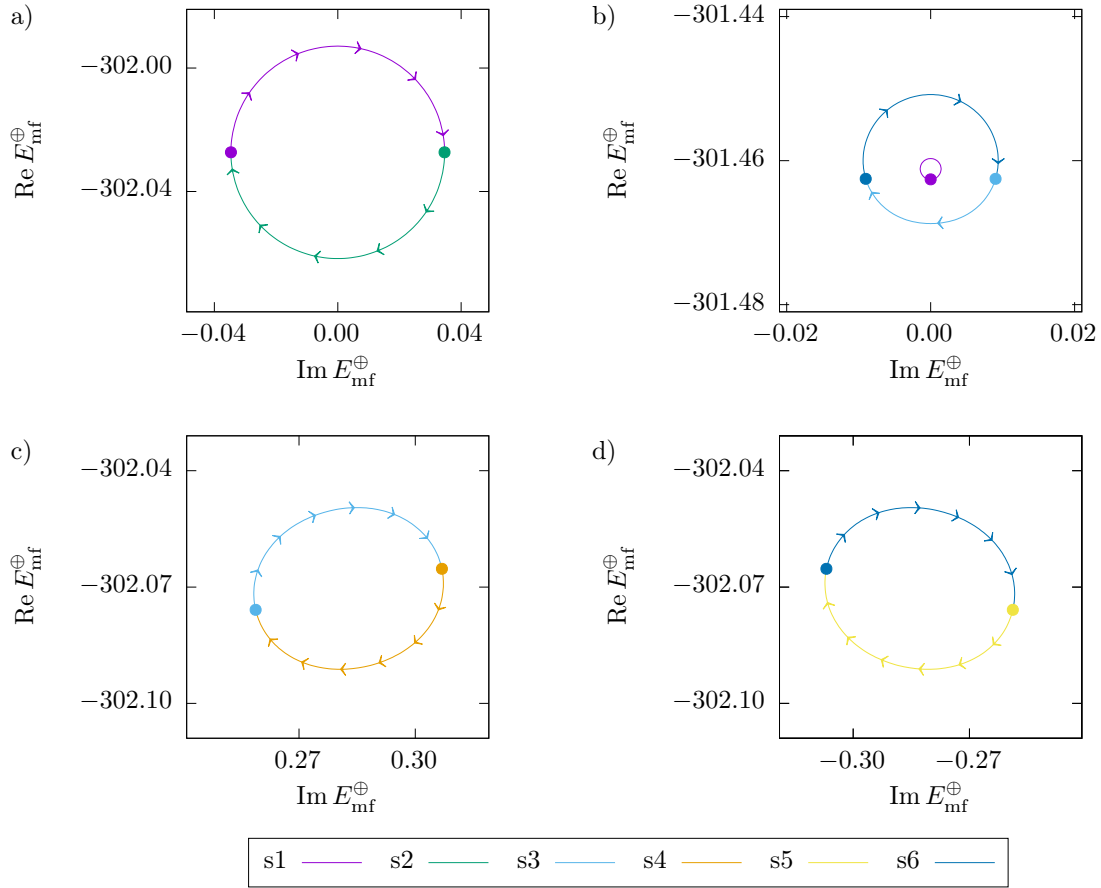
Figure 6. Characteristic exchange of states when an exceptional point is encircled in the complex parameter space. In this case the bifurcation B1 (subfigure (a), $\Gamma_{\text{center}} = 1.03904$, $r = 10^{-3}$), bifurcation B2 (subfigure (b), $\Gamma_{\text{center}} = 0.71967$, $r = 2 \times 10^{-3}$), bifurcation B3a (subfigure (c), $\Gamma_{\text{center}} = 1.10296$, $r = 10^{-4}$) and bifurcation B3b (subfigure (d), $\Gamma_{\text{center}} = 1.10296$, $r = 10^{-4}$) are encircled on the parameter path $\Gamma(x) = \Gamma_{\text{center}} + r e^{\text{j}\phi}$ for $\phi \in [0, 2\pi]$. All plots were calculated for a scattering length of $c_{\text{sc}} = -0.9$.

be examined.

We introduce a new asymmetry parameter in (4), which lifts or lowers the potential wells

$$c_{\text{g},1} = (V_0 + s) + \text{i}\Gamma \text{ and } c_{\text{g},2} = (V_0 - s) + \text{i}\Gamma. \tag{29}$$

The new parameter s allows us to break the remaining trap symmetry of the system, e.g. one potential well is deepend, while the other is flattened. We encircle the bifurcation for the scattering length $c_{\text{sc,crit},2} = -0.93365$ and the coupling parameter $\Gamma = 1.03215$ on the path

$$s = 5 \times 10^{-5} e^{\text{j}\phi} \text{ for } \phi \in [0, 2\pi] \tag{30}$$

and observe a permutation of all five states with each other (see figure 8). Thus we have proven the existence of an exceptional point of order five in this system.

a)

b)

c)

d)

| s1 —— | s2 —— | s3 —— | s4 —— | s5 —— | s6 —— |
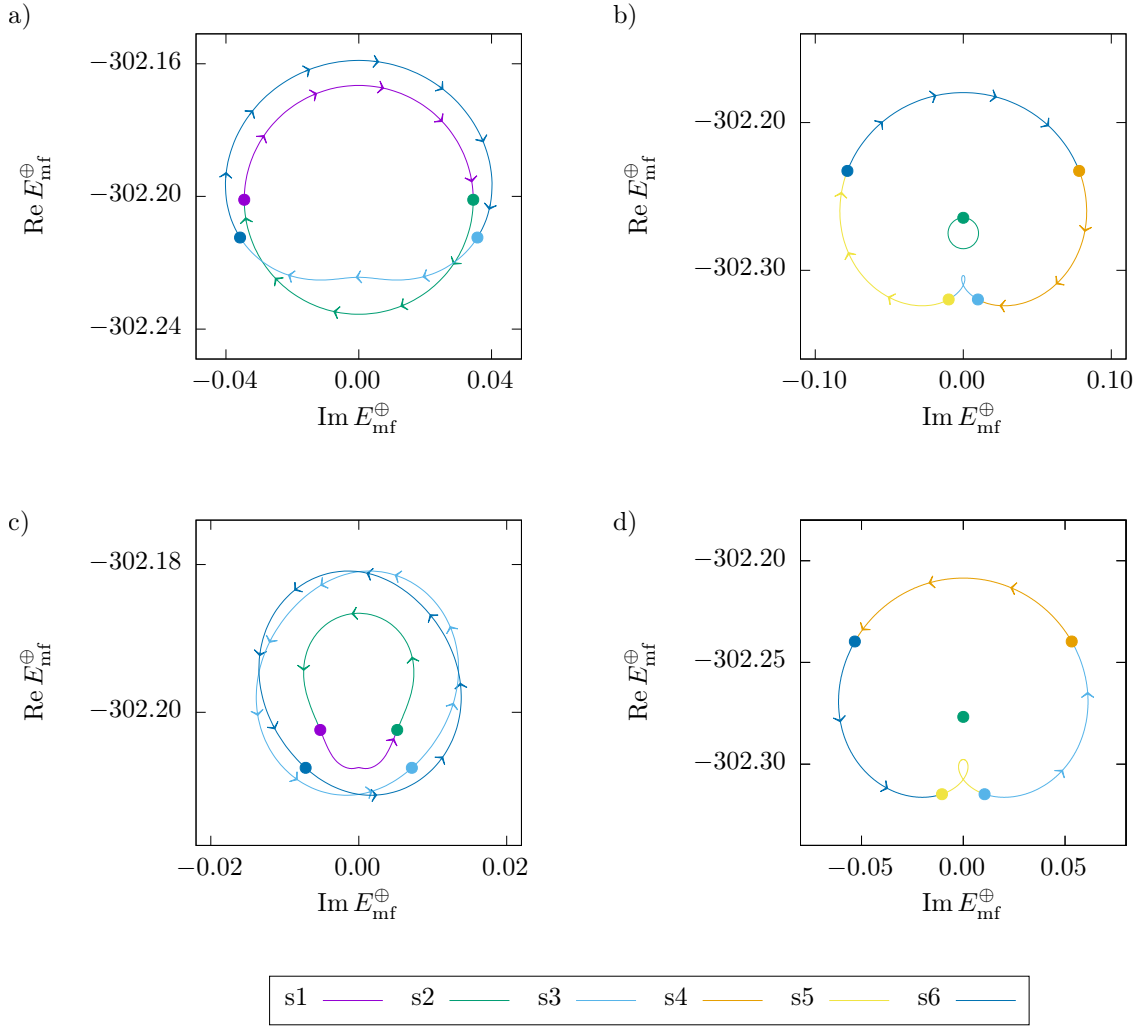
Figure 7. (a) and (c) show the exceptional point at the coalescence of bifurcation B1 and B2 at $c_{\mathrm{sc,crit,1}} = -0.93063$ and $\Gamma = 1.03531$. (b) and (d) show the exceptional point for the merger of the bifurcations B2, B3a and B3b at $c_{\mathrm{sc,crit,2}} = -0.93365$ and $\Gamma = 1.03215$. The exceptional points are encircled in the complex parameter space of Γ (subfigure (a) and (b)) with a radius of $r = 10^{-3}$. In (c) and (d) the exceptional points are encircled in the complex parameter space of c_{sc}. The radius of the circle is $r = 4.65 \times 10^{-4}$ in (c) and $r = 2 \times 10^{-4}$ in (d). One exceptional point (subfigure (b) and (d)) exhibits the full exchange behaviour of a fourth-order exceptional point, in which the original situation is not reached until the point is encircled four times. By contrast the other exceptional point in (a) and (c) has two separated groups of states which only exchange with each other, and therefore the starting condition is already reached again after two full orbits.
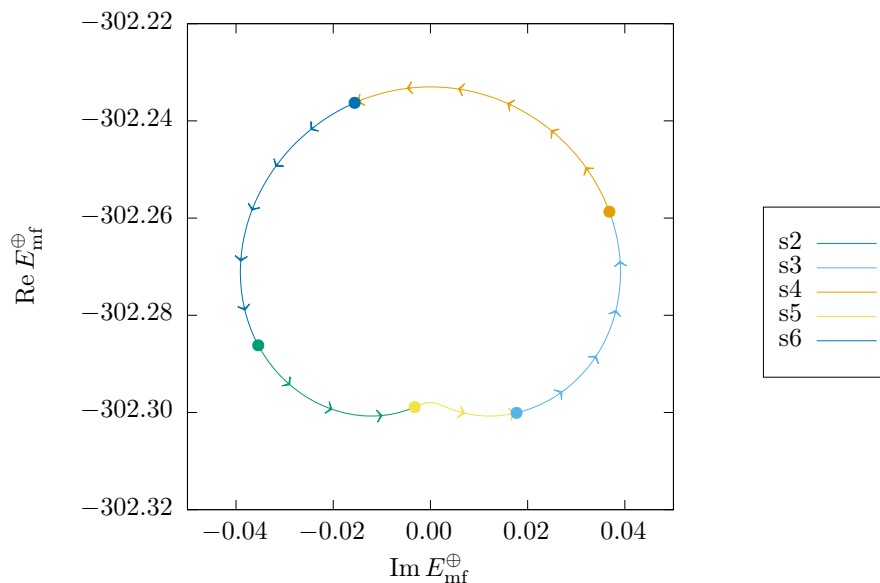
Figure 8. Mean-field energies $E_{\mathrm{mf}}{}^{\oplus}$ when the exceptional point where the bifurcations B2, B3a, and B3b coalesce is encircled in the complex symmetry-parameter space s defined in (30).

5. Summary and conclusion

We have shown how the Gross-Pitaevskii equation for dipolar Bose-Einstein condensates can be analytically continued with an ansatz of coupled Gaussians using bicomplex numbers. Especially the representation in the idempotent basis of the bicomplex numbers can be used to separate the bicomplex equations into twice the number of coupled complex equations. This allows for the reuse of an algorithm developed for the integration of the complex equations.

By solving the analytically continued equations we have shown that a dipolar Bose-Einstein condensate in a $\mathcal{PT}$-symmetric trap exhibits a much richer bifurcation scenario than a condensate without long-range interactions. Most of the properties examined in this paper have revealed interesting mathematical relations in the bicomplex parameter space, which is experimentally inaccessible. However, the understanding of the bifurcation scenrio is important since bifurcations influence crucially the stabiliy of a condensate [64].

Furthermore the additional states, while not experimentally accessible, allow us to examine the order of the exceptional points. Higher-order exceptional points were found in other systems such as EP3s in the hydrogen atom [44] or in Bose-Einstein condensates [49]. Even exeptional points of arbitrary order were shown to exist in a Bose-Hubbard model [46]. However, no EPs with an order higher than three were known for a BEC described by a mean-field model.

With the new (bicomplex) states we have demonstrated the properties of the exceptional points associated with the bifurcations (section 4). In particular the critical

points where two or three bifurcation coalesce were examined. By having shown that a parameter exists for which the encircling of the critical value results in the permutation of all five states participating in the exceptional point, we have indeed proven a fifth-order exceptional point in dipolar Bose-Einstein condensates.

It was shown [65] that the signatures of exceptional points influence the collapse dynamics of Bose-Einstein condensates. With the help of a harmonic inversion analysis [66] the order of degeneracies can be translated into polynomial terms in the time-evolution. Further studies will reveal the influence of such higher-order exceptional points on the collapse dynamics.

References

[1] C. M. Bender, S. Boettcher, and P. N. Meisinger. $\mathcal{PT}$-symmetric quantum mechanics. *J. Math. Phys.*, 40:2201, 1999.

[2] H. F. Jones and Jr. E. S. Moreira. Quantum and classical statistical mechanics of a class of non-Hermitian Hamiltonians. *J. Phys. A*, 43:055307, 2010.

[3] V. Jakubský and M. Znojil. An explicitly solvable model of the spontaneous $\mathcal{PT}$-symmetry breaking. *Czech. J. Phys.*, 55:1113, 2005.

[4] H. Mehri-Dehnavi, A. Mostafazadeh, and A. Batal. Application of pseudo-Hermitian quantum mechanics to a complex scattering potential with point interactions. *J. Phys. A*, 43:145301, 2010.

[5] C. E. Ruter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip. Observation of parity-time symmetry in optics. *Nat. Phys.*, 6:192–195, 2010.

[6] I. V. Barashenkov, G. S. Jackson, and S. Flach. Blow-up regimes in the $\mathcal{PT}$-symmetric coupler and the actively coupled dimer. *Phys. Rev. A*, 88:053817, 2013.

[7] S. Deffner and A. Saxena. Jarzynski equality in $\mathcal{PT}$-symmetric quantum mechanics. *Phys. Rev. Lett.*, 114:150601, 2015.

[8] S. Albeverio, S. Fassari, and F. Rinaldi. The discrete spectrum of the spinless one-dimensional Salpeter Hamiltonian perturbed by δ-interactions. *J. Phys. A*, 48:185301, 2015.

[9] A. Mostafazadeh. Nonlinear spectral singularities for confined nonlinearities. *Phys. Rev. Lett.*, 110:260402, 2013.

[10] S. Bittner, B. Dietz, U. Günther, H. L. Harney, M. Miski-Oglu, A. Richter, and F. Schäfer. $\mathcal{PT}$ Symmetry and Spontaneous Symmetry Breaking in a Microwave Billiard. *Phys. Rev. Lett.*, 108:024101, 2012.

[11] J. Schindler, A. Li, M. C. Zheng, F. M. Ellis, and T. Kottos. Experimental study of active *LRC* circuits with $\mathcal{PT}$ symmetries. *Phys. Rev. A*, 84:040101, 2011.

[12] J. Schindler, Z. Lin, J. M. Lee, H. Ramezani, F. M. Ellis, and T. Kottos. $\mathcal{PT}$-symmetric electronics. *J. Phys. A*, 45:444029, 2012.

[13] A. Ruschhaupt, F. Delgado, and J. G. Muga. Physical realization of $\mathcal{PT}$ -symmetric potential scattering in a planar slab waveguide. *J. Phys. A*, 38:L171, 2005.

[14] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides. Observation of $\mathcal{PT}$-Symmetry Breaking in Complex Optical Potentials. *Phys. Rev. Lett.*, 103:093902, 2009.

[15] H. Ramezani, T. Kottos, R. El-Ganainy, and D. N. Christodoulides. Unidirectional nonlinear $\mathcal{PT}$-symmetric optical structures. *Phys. Rev. A*, 82:043803, 2010.

[16] Z. H. Musslimani, Konstantinos G. Makris, R. El-Ganainy, and D. N. Christodoulides. Optical solitons in PT periodic potentials. *Phys. Rev. Lett.*, 100:30402, 2008.

[17] K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani. $\mathcal{PT}$-Symmetric Periodic Optical Potentials. *Int. J. Theo. Phys.*, 50:1019, 2011.

[18] K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani. Beam Dynamics in $\mathcal{PT}$ Symmetric Optical Lattices. *Phys. Rev. Lett.*, 100:103904, 2008.

[19] K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z.H. Musslimani. $\mathcal{PT}$-symmetric optical lattices. *Phys. Rev. A*, 81:063807, 2010.

[20] K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z.H. Musslimani. Beam dynamics in PT symmetric optical lattices. *Phys. Rev. Lett.*, 100:103904, 2008.

[21] Y. D. Chong, L. Ge, and A. D. Stone. $\mathcal{PT}$-symmetry breaking and laser-absorber modes in optical scattering systems. *Phys. Rev. Lett.*, 106:093902, 2011.

[22] B. Peng, S. Ö. Kaya, F. Lei, F. Minufu, M. Gianfreda, G. L. Long, S. Fan, F. Nori, C. M. Bender, and L. Yang. Parity-time-symmetric whispering-gallery microcavities. *Nat. Phys.*, 10:394–398, 2014.

[23] D. Dast, D. Haag, H. Cartarius, and G. Wunner. Quantum master equation with balanced gain and loss. *Phys. Rev. A*, 90:052120, 2014.

[24] R. Gutöhrlein, J. Schnabel, I. Iskandarov, H. Cartarius, J. Main, and G. Wunner. Realizing $\mathcal{PT}$-symmetric BEC subsystems in closed Hermitian systems. *Journal of Physics A: Mathematical and Theoretical*, 48:335302, 2015.

[25] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell. Observation of Bose-Einstein condensation in a dilute atomic vapor. *Science*, 269:198, 1995.

[26] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet. Evidence of Bose-Einstein condensation in an atomic gas with attractive interactions. *Phys. Rev. Lett.*, 75:1687, 1995.

[27] K. B. Davis, M. O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle. Bose-Einstein condensation in a gas of sodium atoms. *Phys. Rev. Lett.*, 75:3969, 1995.

[28] A. Griesmaier, J. Werner, S. Hensler, J. Stuhler, and T. Pfau. Bose-Einstein Condensation of Chromium. *Phys. Rev. Lett.*, 94:160401, 2005.

[29] L. Santos, G. V. Shlyapnikov, P. Zoller, and M. Lewenstein. Bose-Einstein Condensation in Trapped Dipolar Gases. *Phys. Rev. Lett.*, 85:1791–1794, 2000.

[30] Q. Beaufils, R. Chicireanu, T. Zanon, B. Laburthe-Tolra, E. Maréchal, L. Vernac, J.-C. Keller, and O. Gorceix. All-optical production of chromium Bose-Einstein condensates. *Phys. Rev. A*, 77:061601, 2008.

[31] M. Lu, N. Q. Burdick, S. H. Youn, and B. L. Lev. Strongly dipolar bose-einstein condensate of dysprosium. *Phys. Rev. Lett.*, 107:190401, 2011.

[32] T. Lahaye, J. Metz, B. Fröhlich, T. Koch, M. Meister, A. Griesmaier, T. Pfau, H. Saito, Y. Kawaguchi, and M. Ueda. *d*-Wave Collapse and Explosion of a Dipolar Bose-Einstein Condensate. *Phys. Rev. Lett.*, 101:080401, 2008.

[33] S. Raghavan, A. Smerzi, S. Fantoni, and S. R. Shenoy. Coherent oscillations between two weakly coupled Bose-Einstein condensates: Josephson effects, π oscillations, and macroscopic quantum self-trapping. *Phys. Rev. A*, 59:620–633, 1999.

[34] M. Asad-uz Zaman and D. Blume. Aligned dipolar Bose-Einstein condensate in a double-well potential: From cigar shaped to pancake shaped. *Phys. Rev. A*, 80:053622, 2009.

[35] B. Xiong, J. Gong, H. Pu, W. Bao, and B. Li. Symmetry breaking and self-trapping of a dipolar Bose-Einstein condensate in a double-well potential. *Phys. Rev. A*, 79:013626, 2009.

[36] T. Gericke, P. Würtz, D. Reitz, T. Langen, and H. Ott. High-resolution scanning electron microscopy of an ultracold quantum gas. *Nat. Phys.*, 4:949–953, 2008.

[37] N. P. Robins, C. Figl, M. Jeppesen, and G.R. Dennis. A pumped atom laser. *Nat. Phys.*, 4:731–736, 2008.

[38] D. Schneble, G. K. Campbell, E. W. Streed, M. Boyd, D. E. Pritchard, and W. Ketterle. Raman amplification of matter waves. *Phys. Rev. A*, 69:041601, 2004.

[39] Y. Yoshikawa, T. Sugiura, Y. Torii, and T. Kuga. Observation of superradiant Raman scattering in a Bose-Einstein condensate. *Phys. Rev. A*, 69:041603, 2004.

[40] M. Kreibich, J. Main, H. Cartarius, and G. Wunner. Realizing $\mathcal{PT}$-symmetric non-Hermiticity

with ultracold atoms and Hermitian multiwell potentials. *Phys. Rev. A*, 90:033630, 2014.

[41] S. Ronen, D. C. E. Bortolotti, and J. L. Bohn. Radial and angular rotons in trapped dipolar gases. *Phys. Rev. Lett.*, 98:030406, 2007.

[42] R. Fortanier, D. Dast, D. Haag, H. Cartarius, J. Main, G. Wunner, and R. Gutöhrlein. Dipolar Bose-Einstein condensates in a $\mathcal{PT}$-symmetric double-well potential. *Phys. Rev. A*, 89:063608, 2014.

[43] T. Kato. *Perturbation theory for linear operators.* Springer, Berlin, 1966.

[44] H. Cartarius, J. Main, and G. Wunner. Exceptional Points in Atomic Spectra. *Phys. Rev. Lett.*, 99:173003, 2007.

[45] W. D. Heiss. Chirality of wavefunctions for three coalescing levels. *Journal of Physics A: Mathematical and Theoretical*, 41:244010, 2008.

[46] E. M. Graefe, U. Günther, H. J. Korsch, and A. E. Niederle. A non-Hermitian $\mathcal{PT}$ symmetric Bose-Hubbard model: eigenvalue rings from unfolding higher-order exceptional points. *J. Phys. A*, 41:255206, 2008.

[47] G. Demange and E.-M. Graefe. Signatures of three coalescing eigenfunctions. *J. Phys. A: Math. Theor.*, 45:025303, 2012.

[48] W. D. Heiss. The physics of exceptional points. *Journal of Physics A: Mathematical and Theoretical*, 45:444016, 2012.

[49] W. D. Heiss, H. Cartarius, G. Wunner, and J. Main. Spectral singularities in $\mathcal{PT}$-symmetric Bose–Einstein condensates. *J. Phys. A*, 46:275307, 2013.

[50] R. Gutöhrlein, J. Main, C. Cartarius, and G. Wunner. Bifurcations and exceptional points in dipolar Bose-Einstein condensates. *Journal of Physics A: Mathematical and Theoretical*, 46:305001, 2013.

[51] T. Lahaye, C. Menotti, L. Santos, M. Lewenstein, and T. Pfau. The physics of dipolar bosonic quantum gases. *Reports on Progress in Physics*, 72:126401, 2009.

[52] A. D. McLachlan. A variational solution of the time-dependent Schrodinger equation. *Mol. Phys.*, 8:39–44, 1964.

[53] S. Rau, J. Main, P. Köberle, and G. Wunner. Pitchfork bifurcations in blood-cell-shaped dipolar Bose-Einstein condensates. *Phys. Rev. A*, 81:031605, 2010.

[54] D. Dast, D. Haag, H. Cartarius, J. Main, and G. Wunner. Eigenvalue structure of a Bose-Einstein condensate in a $\mathcal{PT}$-symmetric double well. *J. Phys. A*, 46:375301, 2013.

[55] M.E. Shapiro, D.C. Struppa, and A. Vajiac. Bicomplex Numbers and their Elementary Functions. *Cubo (Temuco)*, 14:61 – 80, 2012.

[56] D. C. Brody and E.-M. Graefe. On complexified mechanics and coquaternions. *Journal of Physics A: Mathematical and Theoretical*, 44:072001, 2011.

[57] D. C. Brody and E.-M. Graefe. Coquaternionic quantum dynamics for two-level systems. *Acta Polytechnica*, 51, 2011.

[58] B. Bagchi and A. Banerjee. Bicomplex hamiltonian systems in quantum mechanics. *Journal of Physics A: Mathematical and Theoretical*, 48:505201, 2015.

[59] R. M. Fortanier. *Variational approaches to dipolar Bose-Einstein condensates.* PhD thesis, Universität Stuttgart, 2014. http://elib.uni-stuttgart.de/opus/volltexte/2014/9223.

[60] B. C. Carlson. Numerical computation of real or complex elliptic integrals. *Numerical Algorithms*, 10:13, 1995.

[61] E.-M. Graefe. Stationary states of a $\mathcal{PT}$ symmetric two-mode Bose–Einstein condensate. *J. Phys. A*, 45:444015, 2012.

[62] H. Cartarius and G. Wunner. Model of a $\mathcal{PT}$-symmetric Bose-Einstein condensate in a δ-function double-well potential. *Phys. Rev. A*, 86:013612, 2012.

[63] D. Dast, D. Haag, H. Cartarius, Günter Wunner, R. Eichler, and J. Main. A Bose-Einstein condensate in a $\mathcal{PT}$-symmetric double well. *Fortschritte der Physik*, 61:124–139, 2013.

[64] D. Haag, D. Dast, A. Löhle, H. Cartarius, J. Main, and G. Wunner. Nonlinear quantum dynamics in a $\mathcal{PT}$-symmetric double well. *Phys. Rev. A*, 89:023601, 2014.

[65] J. Brinker, J. Fuchs, J. Main, G. Wunner, and H. Cartarius. Verification of exceptional points in the collapse dynamics of Bose-Einstein condensates. *Phys. Rev. A*, 91:013609, 2015.

[66] J. Fuchs, J. Main, H. Cartarius, and G. Wunner. Harmonic inversion analysis of exceptional points in resonance spectra. *Journal of Physics A: Mathematical and Theoretical*, 47:125304, 2014.