

Supersymmetric model of a Bose-Einstein condensate in a \mathcal{PT} -symmetric double-delta trap

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Abstract The most important properties of a Bose-Einstein condensate subject to balanced gain and loss can be modelled by a Gross-Pitaevskii equation with an external \mathcal{PT} -symmetric double-delta potential. We study its linear variant with a supersymmetric extension. It is shown that both in the \mathcal{PT} -symmetric as well as in the \mathcal{PT} -broken phase arbitrary stationary states can be removed in a supersymmetric partner potential without changing the energy eigenvalues of the other state. The characteristic structure of the singular delta potential in the supersymmetry formalism is discussed, and the applicability of the formalism to the nonlinear Gross-Pitaevskii equation is analysed. In the latter case the formalism could be used to remove \mathcal{PT} -broken states introducing an instability to the stationary \mathcal{PT} -symmetric states.

Keywords \mathcal{PT} symmetry · supersymmetry · double-delta potential · stationary states

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1 Introduction

Bose-Einstein condensates in a double-well setup, where in one well atoms are extracted from the trap and in the other atoms are added coherently to the condensed phase, have shown to be a good candidate for a realisation

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of a \mathcal{PT} -symmetric quantum system, i.e. a system of which the Hamiltonian commutes with the combined action of the parity and time-reversal operators, $[\mathcal{PT}, H] = 0$ [1,2,3]. These systems are of special interest since they allow for the existence of real eigenvalues despite the presence of gain and loss of the probability amplitude, which is described by the non-Hermitian contributions to the Hamiltonian [4]. Real eigenvalues represent the situation of balanced gain and loss such that a stationary probability distribution in the system of interest exists.

While the existence of \mathcal{PT} -symmetric states has been shown for Bose-Einstein condensates [2,3] there has to be taken special care of the interatomic contact interaction. In the mean-field limit of the Gross-Pitaevskii equation this interaction leads to a nonlinearity in the Hamiltonian. Many aspects of \mathcal{PT} symmetry in quantum systems, e.g. the relation between \mathcal{PT} -symmetric eigenstates and real energy eigenvalues, remain unchanged if the Gross-Pitaevskii nonlinearity $\propto |\psi|^2$ is added [5]. However, the nonlinearity also leads to new features. In linear quantum mechanics usually two energy eigenvalues approach each other when the gain-loss effect is increased until they merge in an exceptional point. For even stronger gain-loss contributions two complex and complex conjugate eigenvalues belonging to \mathcal{PT} -broken wave functions appear. In the nonlinear system these complex eigenvalues are not born in the exceptional point, rather they branch off from one of the real eigenvalues before it vanishes together with the second in the exceptional point. In optical media, where a Kerr nonlinearity leads to a mathematical description equivalent to that of the Gross-Pitaevskii equation, these effects may be exploited for technical applications such as unidirectional wave guides [6] or the propagation of solitons [7,8,9]. However, the \mathcal{PT} -broken states also introduce a dynamical instability to the stationary \mathcal{PT} -symmetric state from which they branch off [10,11].

The formalism of non-relativistic supersymmetry (SUSY) offers an elegant way of removing disturbing \mathcal{PT} -broken states without changing all other states. In an experiment this possibility could be of great benefit. Initially introduced in quantum field theories [12,13,14] supersymmetry has also a large number of applications in non-relativistic quantum mechanics [15,16,17]. A characteristic property is the possibility to relate two quantum mechanical systems with different potentials V_1 and V_2 by a supersymmetric transformation such that they possess almost identical spectra. Apart from the ground state of the system described by V_1 all eigenstates appear also in the system governed by V_2 with exactly the same energies but different wave functions.

Completely new perspectives are offered by supersymmetry in non-Hermitian \mathcal{PT} -symmetric quantum systems. The relations of both symmetries have been studied extensively [18,19,20], where much richer structures than in Hermitian systems and even purely real partner potentials V_2 of complex \mathcal{PT} -symmetric potentials V_1 can be found [21,22,23,24]. In optics the formalism has been used to study theoretically methods of designing the refractive index of optical crystals such that they become unidirectionally invisible [25] or to synthesise desired functionalities [26]. Of particular interest for our pur-

pose is the fact that quantum systems described by a complex \mathcal{PT} -symmetric potential V_1 can be related to a partner potential V_2 in which not only the ground state but any arbitrary state can be removed. Miri et al. [27] have shown that this property can be used to selectively remove unwanted modes from a wave guide without hindering the propagation of desired waves. In this article we want to extend this concept to matter waves.

A simple model that features all effects of a Bose-Einstein condensate in a double-well with balanced gain and loss is the Gross-Pitaevskii equation of the \mathcal{PT} -symmetric double-delta potential [28,29]. It is the main purpose of this paper to perform the first step on the way to remove the \mathcal{PT} -broken states introducing the dynamical instability. To do so, we apply the SUSY formalism to the case of vanishing Gross-Pitaevskii nonlinearity. This potential has often been used to gain deeper insight with analytically accessible energies or wave functions [30,31,32,33,34,35]. We show that the SUSY scheme can indeed be used to remove arbitrary \mathcal{PT} -symmetric and \mathcal{PT} -broken eigenstates. The concept turns out to work well and provides an infinite number of superpotentials for the removal of each eigenstate. To understand the properties of supersymmetry in our system we discuss in detail the characteristics of the singular delta potential, for which so far mathematical investigations in the Hermitian case exist [36,37,38]. Furthermore, we comment on possible extensions of the procedure to nonlinear systems and develop a method of constructing a potential V_2 which for weak nonlinearities leads to good approximate solutions.

The article is organised as follows. In Sect. 2 we introduce the SUSY formalism and apply it to the \mathcal{PT} -symmetric double-delta potential. Then we demonstrate how the procedure can be used to remove an arbitrary eigenstate without influencing the remaining one in Sect. 3. We analyse in particular the case in which two states coalesce at an exceptional point. The applicability of the formalism to systems with a weak nonlinearity is discussed in Sect. 4. Finally we summarise our results and give an outlook on possible extensions of our approach to general nonlinearities in Sect. 5.

2 Supersymmetric extension of the \mathcal{PT} -symmetric double-delta potential

In a first step we investigate how the SUSY formalism acts on the singular and non-Hermitian \mathcal{PT} -symmetric double-delta potential. Since the formalism was set up for linear quantum mechanics we take only into account the linear parts of the Hamiltonian. Then we apply the standard scheme of deriving the SUSY partner \mathcal{H}_2 (i.e. the Fermionic sector in SUSY notation) of a given Hamiltonian \mathcal{H}_1 of the original system (Bosonic sector). To do so, in terms of exact SUSY, the energy of \mathcal{H}_1 is shifted such that the energy of the state we wish to remove is zero. Thus, we consider the one-dimensional Schrödinger equation of the \mathcal{PT} -symmetric double-delta potential in suitable units [28] and

subtract the energy of the current eigenstate,

$$\mathcal{H}_1 \phi_n^{(1)} = \left[-\partial_x^2 + \nu \delta \left(x - \frac{a}{2} \right) + \nu^* \delta \left(x + \frac{a}{2} \right) + \left(\kappa_n^{(1)} \right)^2 \right] \phi_n^{(1)} = 0, \quad (1)$$

where $\phi_n^{(1)}$ is the eigenstate of \mathcal{H}_1 with the corresponding eigenvalue $\mathcal{E}_n^{(1)} = -(\kappa_n^{(1)})^2$. The complex strength of the double-delta potential at $x = \pm a/2$ is denoted by $\nu = -1 + i\gamma$ and $\nu^* = -1 - i\gamma$. The next step is to factorise the Hamiltonian by means of the creation and annihilation operators B^\pm to gain a link between the Bosonic and Fermionic sectors, i.e. \mathcal{H}_1 and its supersymmetric partner Hamiltonian \mathcal{H}_2 , whose eigenstates and eigenenergies shall be calculated. To this end we introduce

$$B^\pm = \mathcal{W}(x) \mp \partial_x \quad (2)$$

with the superpotential $\mathcal{W}(x)$. Using the canonical representation both Hamiltonians can be combined in one SUSY Hamiltonian

$$\begin{aligned} \mathcal{H}_S &= \begin{pmatrix} \mathcal{H}_1 & 0 \\ 0 & \mathcal{H}_2 \end{pmatrix} = \begin{pmatrix} B^+ B^- & 0 \\ 0 & B^- B^+ \end{pmatrix} \\ &= \begin{pmatrix} -\partial_x^2 + \mathcal{W}^2(x) - \mathcal{W}'(x) & 0 \\ 0 & -\partial_x^2 + \mathcal{W}^2(x) + \mathcal{W}'(x) \end{pmatrix}. \end{aligned} \quad (3)$$

Following the relations of quantum mechanical supersymmetry we identify

$$V_1 = \mathcal{W}^2 - \mathcal{W}' \quad (4)$$

with the present double delta-potential appearing in (1). Consequently, we obtain a rule to calculate the superpotential,

$$\mathcal{W}(x) = -\frac{\partial_x \phi_n^{(1)}}{\phi_n^{(1)}}. \quad (5)$$

Using the analytical solutions for $\phi_n^{(1)}$, c.f. Refs. [28,29], the superpotential is given by

$$\mathcal{W} = \begin{cases} -\kappa_n^{(1)} & \text{for } x < -\frac{a}{2}, \\ -\kappa_n^{(1)} \frac{1 + (1 + 2\kappa_n^{(1)}/\nu) \exp(-\kappa_n^{(1)}(2x-a))}{1 - (1 + 2\kappa_n^{(1)}/\nu) \exp(-\kappa_n^{(1)}(2x-a))} & \text{for } -\frac{a}{2} < x < \frac{a}{2}, \\ \kappa_n^{(1)} & \text{for } x > \frac{a}{2}. \end{cases} \quad (6)$$

Equation (6) predicts a jump in the superpotential and therefore a divergence in its first derivative at the positions of the delta functions. Since we use \mathcal{W} and \mathcal{W}' to generate the supersymmetric partner potential V_2 and calculate the eigenfunctions of \mathcal{H}_2 it is necessary to know the impact on the solutions of \mathcal{H}_2 . Hence, we have to understand the appearance of the delta-singularity in detail.

Starting with the stationary Schrödinger equation of the Fermionic sector

$$(\partial_x^2 - [\mathcal{W}^2 + \mathcal{W}']) \phi_n^{(2)} = \mathcal{E}_n^{(2)} \phi_n^{(2)} \quad (7)$$

we find for the jump in the first derivative of the wave function by integrating over a small neighbourhood around the position of the delta function at $x = +a/2$

$$\Delta \partial_x \phi_n^{(2)} = \lim_{\varepsilon \rightarrow 0} \int_{\frac{a}{2}-\varepsilon}^{\frac{a}{2}+\varepsilon} dx \mathcal{W}' \phi_n^{(2)}. \quad (8)$$

These are the only terms which remain in the integral. All other terms of (7) are continuous and vanish in the limit $\varepsilon \rightarrow 0$. Furthermore we can calculate the jump in the derivative of the superpotential from its definition in (5),

$$\begin{aligned} \Delta \mathcal{W} \left(\frac{a}{2} \right) &= \lim_{\varepsilon \rightarrow 0} \int_{\frac{a}{2}-\varepsilon}^{\frac{a}{2}+\varepsilon} dx \partial_x \left[-\frac{\partial_x \phi_n^{(1)}}{\phi_n^{(1)}} \right] \\ &= -\lim_{\varepsilon \rightarrow 0} \left[\frac{\partial_x \phi_n^{(1)} \left(\frac{a}{2} + \varepsilon \right)}{\phi_n^{(1)} \left(\frac{a}{2} + \varepsilon \right)} - \frac{\partial_x \phi_n^{(1)} \left(\frac{a}{2} - \varepsilon \right)}{\phi_n^{(1)} \left(\frac{a}{2} - \varepsilon \right)} \right] \\ &= -\frac{1}{\phi_n^{(1)} \left(\frac{a}{2} \right)} \Delta \partial_x \phi_n^{(1)} \left(\frac{a}{2} \right) = -\nu, \end{aligned} \quad (9)$$

where we used the result for the jump in the derivative of the wave function in a double-delta potential. This result can be used to formally substitute the occurrence of \mathcal{W}' with delta functions. The situation is symmetric, therefore we get a similar result for the integration at $x = -a/2$, namely $\Delta \mathcal{W}(-a/2) = -\nu^*$. By using

$$\mathcal{W}' \rightarrow -\nu \delta \left(x - \frac{a}{2} \right) - \nu^* \delta \left(x + \frac{a}{2} \right), \quad (10)$$

equation (8) assumes the shape

$$\Delta \partial_x \phi_n^{(2)} \left(\frac{a}{2} \right) = -\nu \phi_n^{(2)} \left(\frac{a}{2} \right), \quad \Delta \partial_x \phi_n^{(2)} \left(-\frac{a}{2} \right) = -\nu^* \phi_n^{(2)} \left(-\frac{a}{2} \right). \quad (11)$$

Given all the expressions above, we are able to provide an analytical solution for the partner potential V_2 , viz.

$$\begin{aligned} V_2(x) &= -\nu \delta \left(x - \frac{a}{2} \right) - \nu^* \delta \left(x + \frac{a}{2} \right) \\ &+ \begin{cases} \left(\kappa_n^{(1)} \right)^2 & \text{for } x < -\frac{a}{2}, \\ \left(\kappa_n^{(1)} \right)^2 \left[\frac{1 + (1 + 2\kappa_n^{(1)}/\nu) \exp(-\kappa_n^{(1)}(2x-a))}{1 - (1 + 2\kappa_n^{(1)}/\nu) \exp(-\kappa_n^{(1)}(2x-a))} \right]^2 & \text{for } -\frac{a}{2} < x < \frac{a}{2}, \\ \left(\kappa_n^{(1)} \right)^2 & \text{for } x > \frac{a}{2}. \end{cases} \end{aligned} \quad (12)$$

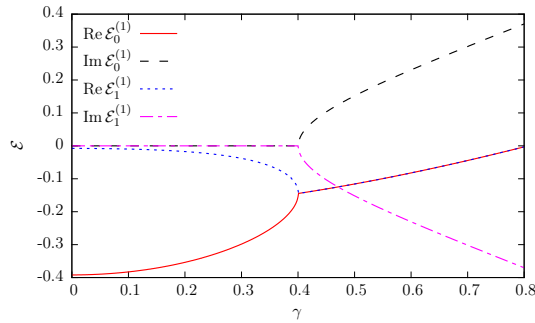


Fig. 1 Energy eigenvalues $\mathcal{E}_n = -(\kappa_n^{(1)})^2$ of the ground state ($n = 0$) and the excited state ($n = 1$). Below a critical value γ_{crit} of the non-Hermiticity parameter γ both energies are real, at γ_{crit} they merge in an exceptional point and above this value of γ both energies are complex and complex conjugate.

3 Removal of arbitrary eigenstates in the linear system

In the previous section we showed that the superpotential and the supersymmetric partner V_2 of the \mathcal{PT} -symmetric double-delta potential V_1 can be found in an analytic way if the corresponding eigenvalue $\kappa_n^{(1)}$ of the system V_1 is known. This eigenvalue has to be determined numerically. To do so, we solve the ordinary differential equation (1) by integrating the wave function outward from $x = 0$ in positive and negative directions. Since its global phase is arbitrary the three initial values $\text{Re } \phi_n^{(1)}(0)$, $\text{Re } \phi_n^{(1)\prime}(0)$, and $\text{Im } \phi_n^{(1)\prime}(0)$ together with an estimate for the complex number $\kappa_n^{(1)}$ are determined in a five-dimensional root search such that physically relevant wave functions are obtained [28]. They have to be normalised (one real condition) and their real and imaginary parts have to vanish in the limits $x \rightarrow \pm\infty$ (four conditions). Numerical solutions of the system V_2 are then found by applying the same technique to the Schrödinger equation

$$\mathcal{H}_2 \phi_n^{(2)} = [-\partial_x^2 + V_2(x)] \phi_n^{(2)} = \kappa_n^{(2)} \phi_n^{(2)} \quad (13)$$

with $V_2(x)$ as defined in (12).

The original system \mathcal{H}_1 shows a typical spectrum of a \mathcal{PT} -symmetric quantum system with two eigenstates, which is shown in Fig. 1. For a purely Hermitian potential, i.e. $\gamma = 0$, we obtain two real energy eigenvalues, which remain real for increasing γ until a critical value $\gamma_{\text{crit}} \approx 0.4005$ is reached. Their wave functions are \mathcal{PT} symmetric. At the critical value both eigenstates merge in an exceptional point, i.e. their energies and wave functions coalesce. Above γ_{crit} the two energies are complex and complex conjugate. The corresponding wave functions are \mathcal{PT} broken. Throughout this article the wave function $\phi_0^{(1)}$ labels the ground state below γ_{crit} and the state with positive imaginary part of the energy above γ_{crit} . The excited state or the state with negative imaginary part of the complex energy are denoted by $\phi_1^{(1)}$. The wave functions $\phi_0^{(1)}$ and $\phi_1^{(1)}$ are drawn in Fig. 2.

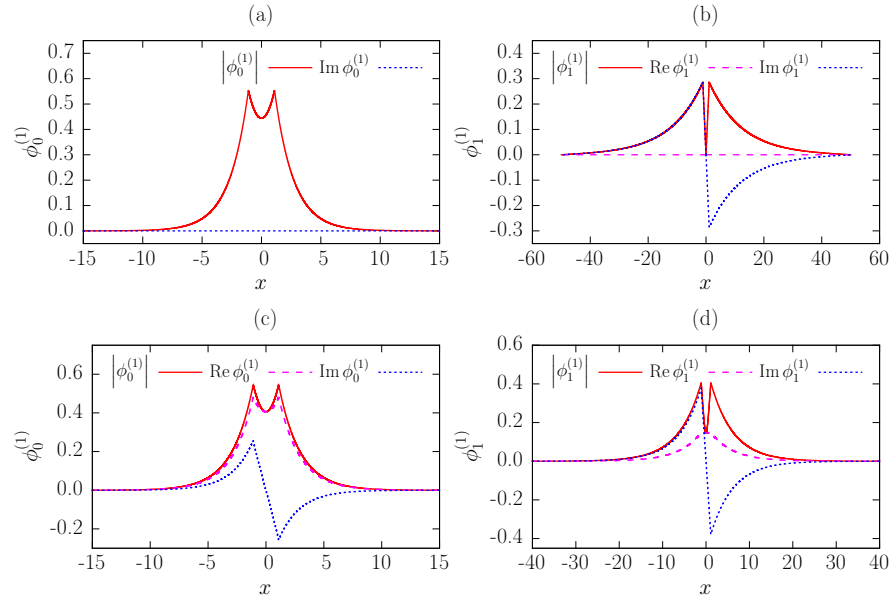


Fig. 2 Wave functions of the ground state $\phi_0^{(1)}$ for $\gamma = 0$ (a), the excited state for $\gamma = 0$ (b), the ground state for $\gamma = 0.3$ (c), and the excited state for $\gamma = 0.3$ (d). Shown are the real and imaginary parts as well as the moduli. In (a) the real part coincides with the modulus and is not shown.

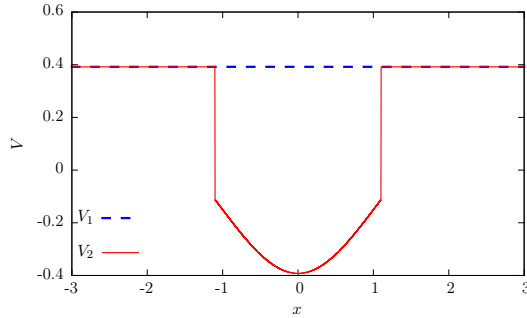


Fig. 3 Potentials V_1 and V_2 for the SUSY formalism applied to the ground state $\phi_0^{(1)}$ in the case $\gamma = 0$.

3.1 Removal of \mathcal{PT} -symmetric states

First we concentrate on the spectrum for values of the non-Hermiticity parameter below the critical value γ_{crit} , which is most illustrative since the eigenvalues $\kappa_n^{(1)}$ remain real and the superpotential \mathcal{W} and the potential V_2 of the Fermion sector can be chosen to preserve \mathcal{PT} symmetry. If we use the ground state $\phi_0^{(1)}$ for the construction of the supersymmetric partner we obtain the potentials V_1 and V_2 , which are shown in Fig. 3 without the singular delta

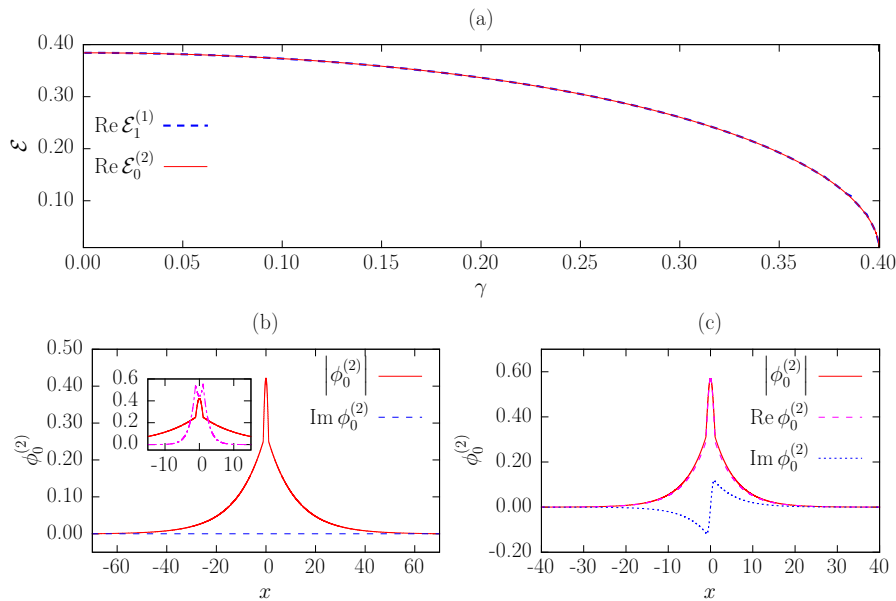


Fig. 4 (a) Energy spectrum $\mathcal{E}_0^{(2)}$ of the only existing state $\phi_0^{(2)}$ in the system described by the potential V_2 for removed ground state. For comparison the energy $\mathcal{E}_1^{(1)}$ of the original system is also shown. (b) Modulus (solid line) and imaginary part (dashed line) of the wave function $\phi_0^{(2)}$ in the case $\gamma = 0$. The real part coincides with the modulus and is not shown. The inset provides a direct comparison of $\phi_0^{(2)}$ (solid line) with the ground state $\phi_0^{(1)}$ of the original system (dashed-dotted line). (c) Modulus (solid line), real (dashed line) and imaginary (dotted line) parts of the wave function $\phi_0^{(2)}$ for $\gamma = 0.3$. In (b) and (c) the deviations from the shapes of the wave functions with the same energy eigenvalues in Figs. 2(b) and (d) are clearly visible.

contributions. The potential V_1 is shifted by the value of the original ground state's energy $\mathcal{E}_0^{(1)} = -(\kappa_0^{(1)})^2$ to be in the case of exact SUSY. More interesting is the shape of V_2 . It contains an additional symmetric potential well between the *repulsive* [cf. the analytic form in (12)] delta functions. This leads to completely different forms of the wave functions.

Due to the SUSY formalism the ground state is removed in the system described by V_2 . The only existing state is the former excited state, which is the ground state of the new system, and hence is labelled $\phi_0^{(2)}$. Its energy must be positive since it must be above that of $\phi_0^{(1)}$, which was set to zero in (1). Additionally we expect $\mathcal{E}_0^{(2)} < |\mathcal{E}_0^{(1)}|$ because the state has to be bound. This is exactly what is found in our numerical solution. Figure 4(a) shows the energy $\mathcal{E}_0^{(2)}$. It is always real, positive, and slightly below $|\mathcal{E}_0^{(1)}|$. As γ approaches γ_{crit} and both states of the original system begin to merge we observe the expected behaviour $\mathcal{E}_0^{(2)} \rightarrow 0$.

The numerical solution for the wave function at $\gamma = 0$ is depicted in Fig. 4(b). The differences to the wave functions with the same energy eigenvalue

of the original system in Figs. 2(b) and (d) are obvious. In particular, the state is a true symmetric ground state of the partner system V_2 , whereas in the original system the state with exactly the same energy eigenvalue was the antisymmetric excited state. Due to the attractive well around the origin the wave function $\phi_0^{(2)}$ has its maximum at $x = 0$ and decays as x increases. The binding energy of this state $|\mathcal{E}_0^{(1)}| - \mathcal{E}_0^{(2)} \approx 0.0077$ is much lower than that of the ground state of the original system, which has the value $|\mathcal{E}_0^{(1)}| \approx 0.3920$. Consequently $\phi_0^{(2)}$ is considerably less localised than the ground state $\phi_0^{(1)}$, which can be seen in the direct comparison in the inset of Fig. 4(b).

As is known from the original \mathcal{PT} -symmetric potential the antisymmetric imaginary part of the ground state's wave function grows in strength for increasing γ . The same behaviour is observed for the ground state $\phi_0^{(2)}$ of the Fermionic sector as can be seen in 4(c). Thus, we observe up to the critical value γ_{crit} the typical behaviour of a \mathcal{PT} -symmetric quantum system with the peculiarity that our potential exhibits only a single bound state.

As was mentioned above the special feature of SUSY in \mathcal{PT} -symmetric quantum systems is the possibility to remove an arbitrary state from the spectrum of the partner potential V_2 provided that its wave function does not have a node. We demonstrate this in our model by removing the energy of the excited state $\phi_1^{(1)}$ for $\gamma \neq 0$. This is achieved with exactly the same procedure as for the ground state with the sole difference that now the eigenvalue $\kappa_1^{(1)}$ is used in the construction of the potential V_2 according to (5) and (7). The spectrum is shown in Fig. 5(a). The potential V_2 for $\gamma = 0.3$ can be seen in Fig. 5(b). This example demonstrates the \mathcal{PT} symmetry of the potential. The partner system can be calculated numerically only up to a minimal value of γ because for $\gamma \rightarrow 0$ the wave function $\phi_1^{(1)}$ approaches more and more the exact shape of the antisymmetric ground state with its node at the origin. This node is reflected in the potential V_2 , which diverges in this limit at $x = 0$. Already for $\gamma = 0.05$ the real part assumes a minimum value of $V_2(0) \approx -540$, which can be observed in Fig. 5(c). This divergence is not surprising since the SUSY formalism is expected to fail for the removal of the excited state in a Hermitian quantum system.

3.2 Removal of \mathcal{PT} -broken states

The construction of a Fermionic sector for our model system in the \mathcal{PT} -broken phase is no difficulty. The partner potential V_2 from (12) remains valid in this case. Only the eigenvalue $\kappa_n^{(1)}$, which appears in the equation, is now complex. An immediate consequence is the loss of \mathcal{PT} symmetry of the partner potential, which is illustrated in Fig. 6, in which the potential V_2 is drawn for the removal of the state $\phi_0^{(1)}$ at $\gamma = 0.5$, i.e. beyond the exceptional point.

The spectrum for the removal of $\phi_0^{(1)}$ on both sides of the exceptional point can be seen in Fig. 7. Also in the \mathcal{PT} -broken phase only one state remains. The

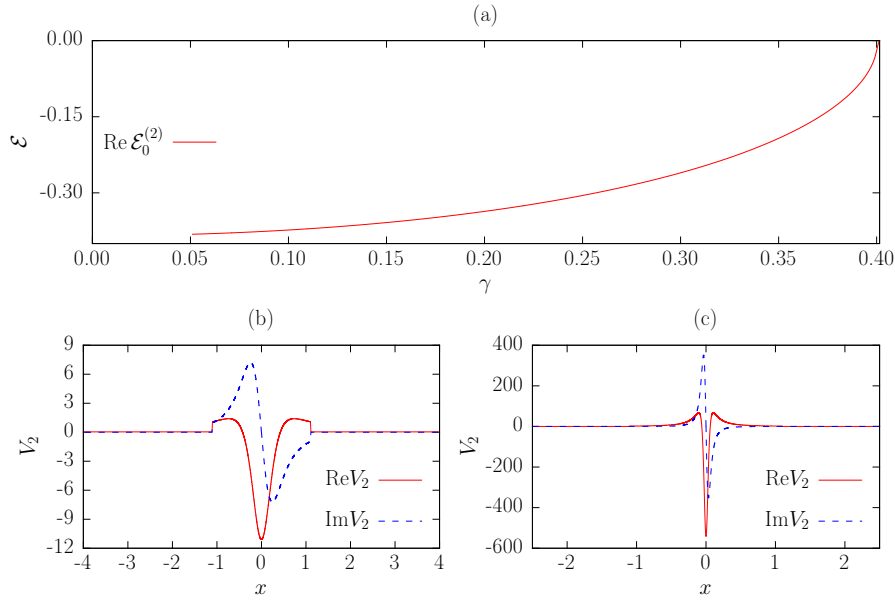


Fig. 5 (a) Energy spectrum $\mathcal{E}_0^{(2)}$ of the only existing state $\phi_0^{(2)}$ in the system described by the potential V_2 for removed excited state up to $\gamma = \gamma_{\text{crit}}$. (b) Real (solid line) and imaginary (dashed line) parts of the potential V_2 for $\gamma = 0.3$. It can be seen that the potential is \mathcal{PT} symmetric. (c) Potential V_2 for $\gamma = 0.05$, where it already begins to diverge.

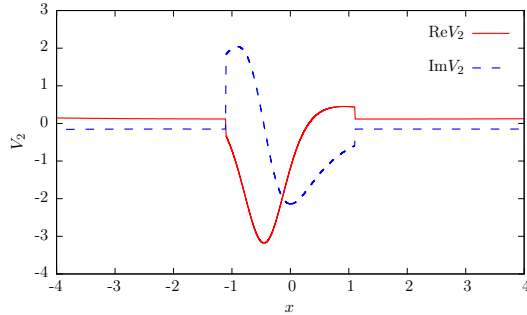


Fig. 6 Real (solid line) and imaginary (dashed line) parts of the potential V_2 for $\gamma = 0.5$ and removed state $\phi_0^{(1)}$ with positive imaginary part of the energy. The potential is no longer \mathcal{PT} symmetric.

purely imaginary energy with $\text{Im } \mathcal{E}_0^{(2)} < 0$ can be understood if one remembers that due to exact SUSY the energy of the original system has been shifted such that the energy of the removed state is set to zero. Thus we expect to observe $\mathcal{E}_0^{(2)} = \mathcal{E}_1^{(1)} - \mathcal{E}_0^{(1)}$ in Fig. 7. This is exactly what is found. In the original system the energies of both eigenstates of the double-delta potential

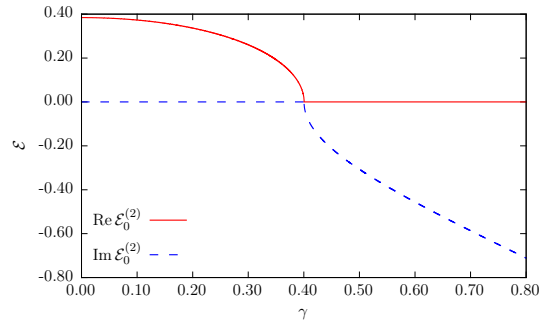


Fig. 7 Real (solid line) and imaginary (dashed line) parts of the energy $\mathcal{E}_0^{(2)}$ for removed state $\phi_0^{(1)}$ on both sides of the exceptional point.

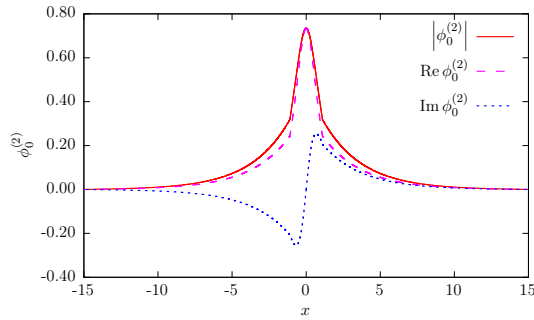


Fig. 8 Modulus (solid line) as well as real (dashed line) and imaginary (dotted line) parts of the wave function $\phi_0^{(2)}$ at the exceptional point at $\gamma_{\text{crit}} \approx 0.4005$.

are complex conjugate and we can calculate

$$\mathcal{E}_0^{(2)} = \mathcal{E}_1^{(1)} - \mathcal{E}_0^{(1)} = 2i \text{Im} \mathcal{E}_1^{(1)} \quad (14)$$

with $\text{Im} \mathcal{E}_1^{(1)} = -\text{Im} \mathcal{E}_0^{(1)} < 0$.

3.3 Behaviour at the exceptional point

In the spectrum in Fig. 7 we also observe that $\mathcal{E}_0^{(2)} = 0$ at γ_{crit} . This is not surprising since at the exceptional point both energies of the original system coincide. However, there remains one question. The SUSY formalism – as it is introduced in Hermitian quantum mechanics – can be used to remove the ground state of \mathcal{H}_1 . Since exactly at the exceptional point also the original \mathcal{PT} -symmetric double-delta potential V_1 has only *one* linearly independent state there should exist *no* wave function at $\gamma = \gamma_{\text{crit}}$. However, this is not the case. Numerically we find a solution at the critical value of γ , which is \mathcal{PT} symmetric. Its wave function is shown in Fig. 8.

There are two possible interpretations of this fact. Firstly, we may assume that the supersymmetry formalism fails in the construction of a true Fermionic sector if the potential V_1 exhibits coalescing eigenstates. Secondly, we may interpret the coalescence at the exceptional point as two individual wave functions which are just equal. Then one may argue that one of these wave functions vanishes, whereas the second survives in the Fermionic sector and supersymmetry is broken. For this interpretation one has to circumvent the difficulty that for broken supersymmetry no state with $\mathcal{E}_n^{(2)} = 0$ may exist, which can be achieved by giving up the energy shift required for exact SUSY. However, both possibilities are only interpretations which as a matter of principle cannot be distinguished. The important fact is that, independently from the state $\phi_n^{(1)}$ which is removed, the potential V_2 *always* exhibits *one* eigenstate, even at the exceptional point.

3.4 Infinitely many superpotentials and real eigenvalues in non- \mathcal{PT} -symmetric potentials

In Sect. 2 we introduced the superpotential $\mathcal{W}(x)$ in the standard form shown in (5). However, this is only one possible solution of the differential equation (4). Its solutions possess an arbitrary integration constant. We mentioned above that in the \mathcal{PT} -symmetric phase of the Bosonic sector \mathcal{H}_1 the potential V_2 of the Fermionic sector preserves \mathcal{PT} symmetry. This is the case because the \mathcal{PT} -symmetric wave function $\phi_n^{(1)}$ of \mathcal{H}_1 is used in the construction of $\mathcal{W}(x)$ in (5), which chooses the integration constant appropriately. In general, V_2 will not be \mathcal{PT} symmetric. Since every \mathcal{H}_2 must be isospectral with that chosen according to the standard form (5) it will possess only one eigenstate with *real* energy in spite of the fact that it is neither Hermitian nor \mathcal{PT} symmetric. Thus, the exploitation of the freedom of the integration constant offers one possibility to construct Hamiltonians which do not possess a special symmetry but exhibit real eigenvalues and even purely real spectra. This finding is equivalent to that discussed in [27].

Integration of the differential equation (4) with the potential

$$V_1(x) = -\mathcal{E}_1 = \left(\kappa_n^{(1)}\right)^2 \quad (15)$$

outside and inside the delta functions leads to the superpotential

$$\mathcal{W}(x) = -\kappa_n^{(1)} \tanh\left(\kappa_n^{(1)}(x - \xi)\right), \quad (16)$$

where ξ is a complex integration constant. One of these solutions is always found if the differential equation (4) is solved numerically. Note that the form presented in Eqs. (6) and (12) corresponds to $\xi \rightarrow \infty$ and $\xi \rightarrow -\infty$ in the intervals $x > a/2$ and $x < -a/2$, respectively.

An example is given in Fig. 9, in which the superpotential, the potential V_2 and the wave function obtained with arbitrary choices for ξ in the intervals

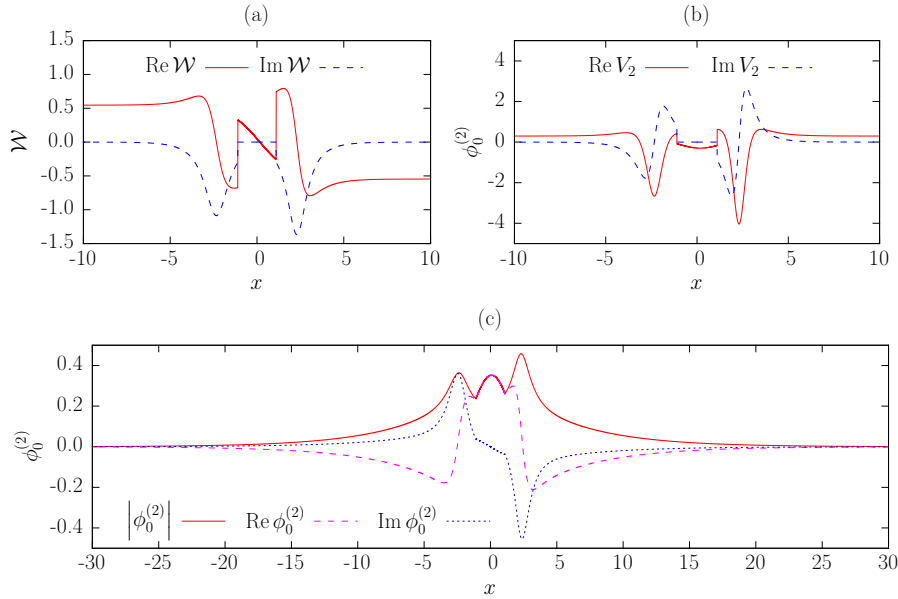


Fig. 9 (a) Real (solid line) and imaginary (dashed line) parts of the superpotential for the removal of the ground state and $\gamma = 0.3$. The values of the integration constant ξ are $\xi = 2.30 + 2.18i$ and $\xi = -2.34 + 2.02i$ in the intervals $x > a/2$ and $x < -a/2$, respectively. (b) Potential V_2 for the same case. (c) Wave functions of the eigenstate $\phi_0^{(2)}$ obtained with this potential.

to the left and right of the delta functions are shown for the case $\gamma = 0.3$ and the removal of the ground state of \mathcal{H}_1 . One clearly recognises that neither V_2 nor the wave function $\phi_0^{(2)}$ are \mathcal{PT} symmetric. Nevertheless, the spectrum obtained in this way is identical to that shown in Fig. 7.

4 Extension to systems with a weak Gross-Pitaevskii nonlinearity

The Gross-Pitaevskii equation contains a nonlinearity which we did not consider so far. In the cold and dilute gas forming a Bose-Einstein condensate the van der Waals interaction can be described correctly by an s-wave scattering process, and the relevant physical parameter defining the strength of the interaction is given by the s-wave scattering length a . Since this value can be adjusted close to Feshbach resonances by shifting the molecular energy levels with an external magnetic field it can be chosen arbitrarily small. However, it is unlikely that the nonlinearity can be set exactly to zero. Every experiment will at least be affected by small perturbations. On the other hand it is also unlikely that the SUSY procedure for the construction of the Fermionic sector's potential will work in the nonlinear system since the formalism relies on the linearity of the Hamiltonian. It is the purpose of this section to show that still good approximate results can be obtained.

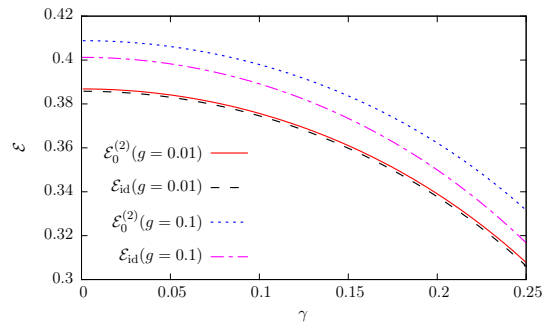


Fig. 10 Energies $\mathcal{E}_0^{(2)}$ for the nonlinearity parameters $g = 0.1$ and $g = 0.01$ in comparison with the ideal values \mathcal{E}_{id} according to (18). For both values of g the actual energies deviate slightly from the expectation, however, the difference is small and the SUSY formalism remains applicable with a reasonable quality of the results.

Including the Gross-Pitaevskii nonlinearity, the potential V_1 required in the SUSY formalism is given by

$$V_1(x) = \left(\kappa_n^{(1)}\right)^2 + \nu\delta\left(x - \frac{a}{2}\right) + \nu^*\delta\left(x + \frac{a}{2}\right) + g\left|\phi_n^{(1)}\right|^2 \quad (17)$$

with the nonlinearity parameter $g \propto a$. In the units given a small nonlinearity means $g \ll 1$. The usage of this V_1 in the differential equation (4) will clearly not lead to a correct partner system \mathcal{H}_2 . With this ansatz the nonlinearity will enter into V_2 with the shape of $\phi_n^{(1)}$ and not with that of the solution $\phi_n^{(2)}$ of the Fermionic sector. Despite this fact the ansatz can be used for approximate solutions as can be seen in Fig. 10. In this example the superpotential \mathcal{W} was calculated with the differential equation (4) for the removal of the ground state and two different small nonlinearities $g \leq 0.1$. The comparison between the energy $\mathcal{E}_0^{(2)}$ and the ideal value

$$\mathcal{E}_{\text{id}} = \mathcal{E}_1^{(1)} - \mathcal{E}_0^{(1)} \quad (18)$$

is small but noticeable.

The method used for constructing the superpotential contains a freedom of one integration constant as explained in Sect. 3.4. In the nonlinear system this constant is no longer arbitrary. It influences the energies $\mathcal{E}_n^{(2)}$. We tried to exploit this freedom to improve the results of the energies or even to enforce the equality $\mathcal{E}_0^{(2)} = \mathcal{E}_{\text{id}}$. We found that the values shown in Fig. 10 cannot be improved further. Some choices of the integration constant even lead to completely asymmetric potentials V_2 . However, the important finding of this section is that the nonlinearity obviously does not completely destroy the concept. In the case of small g the approximation works reasonably well. The energies $\mathcal{E}_0^{(2)}$ are always slightly above \mathcal{E}_{id} , but in particular for $g = 0.01$ an almost unchanged energy is obtained.

5 Summary and outlook

In this paper we studied the supersymmetric extension of the \mathcal{PT} -symmetric double-delta potential. It was possible to show that the SUSY formalism from non-relativistic quantum mechanics can be used to remove any of the two states of the original system in an adequately chosen supersymmetric partner potential. The second state is present in the new system with exactly the same energy as in the original system but a different wave function. It is always a symmetric ground state since the original double-delta potential exhibits only two bound solutions. The application of the formalism to both states is possible because all solutions of the non-Hermitian \mathcal{PT} -symmetric system ($\gamma \neq 0$) are nodeless. In the Hermitian double-delta potential the excited state exhibits a node at the origin and cannot be removed from the spectrum. The corresponding potential V_2 diverges, which could be shown by reducing the non-Hermiticity in our model potential.

Even exactly at the exceptional point, where only one state is present in the system, the formalism can be applied. It leads to one wave function in the partner system with the correct energy. In principle, infinitely many superpotentials and hence also potentials of the Fermionic sector can be found. This freedom can be used to either find partner potentials which preserve the relations of \mathcal{PT} symmetry or lead to cases in which a non-Hermitian non- \mathcal{PT} -symmetric potential exhibits real eigenvalues or even purely real spectra.

In an extension we investigated whether the formalism can also be used for the nonlinear Gross-Pitaevskii equation. We found that for small nonlinearities a partner potential can be constructed, of which the remaining state's energy is almost unchanged in comparison with its counterpart in the original system, whereas the other has vanished. Thus, the results are very similar to the linear case and the most important feature of the supersymmetry concept is preserved in the nonlinear system. One state is removed from the spectrum without disturbing the others too much. Unfortunately this is not true for stronger nonlinearities. Here the simple construction of a superpotential with a procedure adapted from linear quantum mechanics will fail. However, the case of strong nonlinearities is the most interesting for the application of the formalism. It is desirable to remove the \mathcal{PT} -broken states branching off from one of the \mathcal{PT} -symmetric eigenstates and introducing a dynamical instability.

Certainly a way to extend the formalism to arbitrary strengths of the non-linearity is the greatest challenge for future work. For this purpose it could be useful to investigate the many-particle description of the condensate in second quantisation. It could be promising to try to factorise the Hamiltonian such that generalised creation and annihilation operators suitable for the supersymmetry concept can be introduced. It will be interesting to see whether then two supersymmetric partner systems can be created and how their mean-field limits are related to each other.

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